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Vepstas Representation for ζ(s)

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Abstract: We consider the Riemann zeta function for positive integers, which allows realize a binomial inversion to obtain the coefficients in the corresponding Vepstas representation. We show that $\zeta'(-1)$ implies an identity involving harmonic numbers and the Glaisher-Kinkelin's constant.

Key words: Riemann zeta function - Harmonic numbers - Euler-Mascheroni's constant

INTRODUCTION

Vepstas [1] showed the following representation for the Riemann zeta function [2]:

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\sum_{k=0}^{\infty}(-1)^k \binom{s-1}{k} a_k\,, s \neq 1; \qquad (1)$$

But we have the relation:

$$s\binom{s-1}{k} = (k+1)\binom{s}{k+1}, \qquad (2)$$

Therefore:

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - \sum_{k=0}^{\infty}(-1)^k (k+1)\binom{s}{k+1} a_k, \qquad (3)$$

Then:

$$\zeta'(s) = -\frac{1}{(s-1)^2} - \sum_{k=0}^{\infty}(-1)^k (k+1)\frac{d}{ds}\binom{s}{k+1}\cdot a_k\,. \qquad (4)$$

We know the properties [2-4]:

$$\left[\frac{d}{ds}\binom{s}{k+1}\right]_{s=-1} = (-1)^k H_{k+1}\,, \zeta'(-1) = \frac{1}{12} - \ln A\,, \qquad (5)$$

Involving harmonic numbers and the Glaisher-Kinkelin's constant $A = 1.2824\,2712\,91\ldots$, thus from (4) and (5):

$$\sum_{k=0}^{\infty}(k+1)\,a_k\,H_{k+1} = \ln A - \frac{1}{3}. \qquad (6)$$

Similarly, (4) and the expressions [2, 4]:

$$\left[\frac{d}{ds}\binom{s}{k+1}\right]_{s=0} = \frac{(-1)^k}{k+1}\,, \zeta'(0) = -\ln\sqrt{2\pi}\,, \qquad (7)$$

Imply the identity [1]:

$$\sum_{k=0}^{\infty} a_k = \ln\sqrt{2\pi} - 1\,. \qquad (8)$$

Now we consider (1) for $s = n+1, n \geq 1$, then:

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$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}a_k=\frac{1}{n+1}\left[\frac{n+2}{2\,n}-\zeta(n+1)\right]-\frac{1}{2}+\gamma_0\,, \tag{9}$$

Such that $\gamma_0 = 0.5772\,1566\,4901\,\ldots$ is the Euler-Mascheroni's constant and $a_0=\frac{1}{2}-\gamma_0$. The binomial inversion [3, 5, 6] of (9) is given by:

$$a_n=\frac{2n+1}{2(n+1)}-\gamma_0+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left[\frac{1}{k}-\frac{\zeta(k+1)}{k+1}\right], \tag{10}$$

where it was applied the relation [7]:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}=-\frac{n}{n+1}. \tag{11}$$

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