Notes on Motivic Cohomology

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Introduction

This book was written by Carlo Mazza and Charles Weibel on the basis of the lectures on the motivic cohomology which I gave at the Institute for Advanced Study in Princeton in 1999/2000.

From the point of view taken in these lectures, motivic cohomology with coefficients in an abelian group A is a family of contravariant functors

$$H^{p,q}(-,A) : Sm/k \to Ab$$

from smooth schemes over a given field k to abelian groups, indexed by integers p and q. The idea of motivic cohomology goes back to P. Deligne, A. Beilinson and S. Lichtenbaum.

Most of the known and expected properties of motivic cohomology (predicted in [ABS87] and [Lic84]) can be divided into two families. The first family concerns properties of motivic cohomology itself – there are theorems concerning homotopy invariance, Mayer-Vietoris and Gysin long exact sequences, projective bundles, etc. This family also contains conjectures such as the Beilinson-Soule vanishing conjecture ($H^{p,q} = 0$ for $p < 0$) and the Beilinson-Lichtenbaum conjecture, which can be interpreted as a partial étale descent property for motivic cohomology. The second family of properties relate motivic cohomology to other known invariants of algebraic varieties and rings. The power of motivic cohomology as a tool for proving results in algebra and algebraic geometry lies in the interaction of the results in these two families; specializing general theorems about motivic cohomology to the cases when they may be compared to classical invariants, one gets new results about these invariants.

The idea of these lectures was to define motivic cohomology and to give careful proofs for the elementary results in the second family. In this sense they are complimentary to the study of [VSF00], where the emphasis is on the properties of motivic cohomology itself. In the process, the structure of the proofs forces us to deal with the main properties of motivic cohomology as well (such as homotopy invariance). As a result, these lectures cover a considerable portion of the material of [VSF00], but from a different point of view.

One can distinguish the following "elementary" comparison results for motivic cohomology. Unless otherwise specified, all schemes below are assumed to be smooth or (in the case of local or semilocal schemes) limits of smooth schemes.

1. $H^{p,q}(X,A)=0$ for $q<0$, and for a connected X one has

$$H^{p,0}(X,A)=\begin{cases} A & \text{for } p=0 \\ 0 & \text{for } p\neq 0 \end{cases}$$

2. one has

$$H^{p,1}(X,\mathbb{Z})=\begin{cases} \mathcal{O}^*(X) & \text{for } p=1 \\ Pic(X) & \text{for } p=2 \\ 0 & \text{for } p\neq 1,2 \end{cases}$$

3. for a field k, one has $H^{p,p}(Spec(k),A)=K_p^M(k)\otimes A$ where $K_p^M(k)$ is the p-th Milnor K-group of k (see [Mil70]).

4. for a strictly henselian local scheme S over k and an integer n prime to char(k), one has

$$H^{p,q}(S,\mathbb{Z}/n)=\begin{cases} \mu_n^{\otimes q}(S) & \text{for } p=0 \\ 0 & \text{for } p\neq 0 \end{cases}$$

   where $\mu_n(S)$ is the groups of n-th roots of unity in S.

5. one has $H^{p,q}(X,A)=CH^q(X,2q-p;A)$. Here $CH^i(X,j;A)$ denotes the higher Chow groups of X introduced by S. Bloch in [Blo86], [Blo94]. In particular,

$$H^{2q,q}(X,A)=CH^q(X)\otimes A,$$

   where $CH^q(X)$ is the classical Chow group of cycles of codimension q modulo rational equivalence.

The isomorphism between motivic cohomology and higher Chow groups leads to connections between motivic cohomology and algebraic K-theory, but we do not discuss these connections in the present lectures. See [Blo94], [BL94], [FS00], [Lev98] and [SV00].

Deeper comparison results include the theorem of M. Levine comparing $CH^i(X, j; \mathbb{Q})$ with the graded pieces of the gamma filtration in $K_*(X) \otimes \mathbb{Q}$ [Lev94], and the construction of the spectral sequence relating motivic cohomology and algebraic K-theory for arbitrary coefficients in [BL94] and [FS00].

The lectures in this book may be divided into two parts, corresponding to the fall and spring terms. The fall term lectures contain the definition of motivic cohomology and the proofs for all of the comparison results listed above except the last one. The spring term lectures contain more advanced results in the theory of sheaves with transfers and the proof of the final comparison result (5).

The definition of motivic cohomology which is used here goes back to the work of Andrei Suslin in about 1985. As far as I understand, when he came up with this definition he was able to prove the first three of the comparison results stated above. In particular the proof of the comparison (3) between motivic cohomology and Milnor's K-groups given in these lectures is exactly Suslin's original proof. The proofs of the last two comparison results (4) and (5) are also based on results of Suslin. Suslin's formulation of the Rigidity Theorem ([Sus83]; see Theorem 7.20) is a key result needed for the proof of (4), and Suslin's moving lemma (Theorem 18A.1 below) is a key result needed for the proof of (5).

It took ten years and two main new ideas to finish the proofs of the comparisons (4) and (5). The first one, which originated in the context of the qfh-topology and was later transferred to sheaves with transfers (definition 2.1), is that the sheaf of finite cycles $\mathbb{Z}_{tr}(X)$ is the *free* object generated by X. This idea led to a group of results, the most important of which is lemma 6.23. The second idea, which is the main result of [CohTh], is represented here by theorem 13.7. Taken together they allow one to efficiently do homotopy theory in the category of sheaves with transfers.

A considerable part of the first half of the lectures is occupied by the proof of (4). Instead of stating it in the form used above, we prove a more detailed theorem. For a given weight q, the motivic cohomology groups $H^{p,q}(X, A)$ are defined as the hypercohomology (in the Zariski topology) of X with coefficients in a complex of sheaves $A(q)|_{X_{Zar}}$. This complex is the restriction to the small Zariski site of X (i.e., the category of open subsets of X) of a complex $A(q)$ defined on the site of all smooth scheme over k with the Zariski and even the étale topology. Restricting $A(q)$ to the small étale site of X, we may consider the étale version of motivic cohomology,

$$H_L^{p,q}(X, A) := \mathbb{H}_{et}^p(X, A(q)|_{X_{et}}).$$

The subscript L is in honor of Steve Lichtenbaum, who first envisioned this construction in [Lic94].

Theorem 10.2 asserts that the étale motivic cohomology of any X with coefficients in $\mathbb{Z}/n(q)$ where n is prime to $\mathrm{char}(k)$ are isomorphic to $H_{et}^p(X, \mu_n^{\otimes q})$. This implies the comparison result (4), since the Zariski and the étale motivic cohomology of a strictly henselian local scheme X agree. There should also be analog of (4) for the case of $\mathbb{Z}/\ell^r$ coefficients where $\ell = \mathrm{char}(k)$, involving the logarithmic de Rham-Witt sheaves $\nu_r^q[-q]$, but I do not know much about it. We refer the reader to [GL00] for more information.

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Introduction to the second part.

The main goals of the second part are to introduce the triangulated category of motives, and to prove the final comparison theorem (5). Both require an understanding of the cohomological properties of sheaves associated with homotopy invariant presheaves with transfers for the Zariski and Nisnevich topologies. This is addressed in lectures 11, 12 and 13.

A crucial role will be played by theorem 13.7: if F is a homotopy invariant presheaf with transfers, and k is a perfect field, then the associated Nisnevich sheaf F_{Nis} is homotopy invariant, and so is its cohomology. For reasons of exposition, the proof of this result is postponed and occupies lectures 20 to 23.

In lectures 14 and 15 we introduce the triangulated category of motives $\mathbf{DM}^{\text{eff.}-}_{Nis}(k, R)$ and study its basic properties. In particular we give a projective bundle theorem (15.12) and show that the product on motivic cohomology (defined in 3.11) is graded-commutative.

Lectures 16 to 19 deal with equidimensional algebraic cycles, leading up to the proof of the final comparison theorem 19.1: for any smooth separated scheme X over a perfect field k, we have

$$H^{p,q}(X, \mathbb{Z}) \cong CH^q(X, 2q - p).$$

The proof relies on three intermediate results. First we show (in 16.7) that the motivic complex $\mathbb{Z}(i)$ is quasi-isomorphic to the Suslin-Friedlander chain complex $\mathbb{Z}^{SF}(i)$, which is built using equidimensinal cycles; our proof of this requires the field to be perfect. Then we show (in 17.20) that Bloch's higher Chow groups are presheaves with transfers over any field. The final ingredient is a result of Suslin (18.3) comparing equidimensional cycles to higher Chow groups over any affine scheme.

The final lectures (20 to 23) are dedicated to the proof of 13.7. Using technical results from lecture 20, we first prove (in 21.3) that F_{Nis} is homotopy invariant. The proof that its cohomology is homotopy invariant occupies most of lecture 23. We conclude with a proof that the sheaf F_{Nis} admits a "Gersten" resolution.

Contents

Lecture 1

The category of finite correspondences

In this lecture we shall define the additive category Cor_k of finite correspondences over a field k. The objects of Cor_k will be the smooth separated schemes (of finite type) over k. The morphisms in Cor_k from X to Y will be the finite correspondences, which are special kinds of cycles in $X \times Y$. Composition is defined so that Cor_k contains the category Sm/k of smooth separated schemes over k.

By convention, all schemes will be separated, and defined over k. Although smooth schemes always have finite type over k [EGA4, 17.3.1], we will sometimes refer to local and even semi-local schemes as being smooth; by this we mean that they are the local (resp., semi-local) schemes associated to points on a smooth scheme.

Our point of view will be that a cycle in a scheme T is a formal $\mathbb{Z}$-linear combination of irreducible closed subsets of T. Each irreducible closed subset W is the support of its associated integral subscheme $\tilde{W}$ so W and $\tilde{W}$ determine each other. Thus we can ascribe some algebraic properties to W. We say that W is *finite* along a morphism $T \to S$ if the restriction $\tilde{W} \to S$ is a finite morphism. A cycle $\sum n_i W_i$ is said to be finite along a morphism if each W_i is finite.

Definition 1.1. If X is a smooth connected scheme over k, and Y is any (separated) scheme over k, an **elementary correspondence** from X to Y is an irreducible closed subset W of $X \times Y$ whose associated integral subscheme is finite and surjective over X. By an elementary correspondence from a non-connected scheme X to Y, we will mean an elementary correspondence from a connected component of X to Y.

The group $Cor(X,Y)$ is the free abelian group generated by the elementary correspondences from X to Y. The elements of $Cor(X,Y)$ will be called **finite correspondences**.

If X is not connected and $X = \cup X_i$ is the decomposition into its connected components, our definition implies that $Cor(X,Y) = \oplus_i Cor(X_i,Y)$.

Example 1.2. Let $f : X \to Y$ be a morphism in Sm/k. If X is connected, the graph Γ_f of f is an elementary correspondence from X to Y. If X is not connected, the sum of the components of Γ_f is a finite correspondence from X to Y. Indeed the projection $\Gamma_f \to X$ is an isomorphism, and Γ_f is closed because Y is separated over k.

The graph Γ_1 of the identity on X is the support of the diagonal $\Delta(X) \subset X \times X$. We will write id_X for the finite correspondence Γ_1 from X to itself. It will be the identity element of $Cor(X,X)$ for the composition product. Note that id_X is an elementary correspondence when X is integral.

If X is connected, Y is smooth and $f : X \to Y$ is finite and surjective, the transpose of Γ_f in $Y \times X$ is a finite correspondence from Y to X. This is a useful construction; see exercise 1.11 below for one application.

Construction 1.3. *Every subscheme Z of $X \times Y$ which is finite and surjective over X determines a finite correspondence $[Z]$ from X to Y.*

Proof. If Z is integral then its support $[Z]$ is by definition an elementary correspondence. In general we associate to Z the finite correspondence $\sum n_i W_i$, where the W_i are the irreducible components of the support of Z which are surjective over a component of X and n_i is the geometric multiplicity of W_i in Z, i.e., the length of the local ring of Z at W_i (See [Ser65] or [Ful84]). □

We will now define an associative and bilinear composition for finite correspondences between smooth schemes. For this, it suffices to define the composition $W \circ V$ of elementary correspondences $V \in Cor(X,Y)$ and $W \in Cor(Y,Z)$. Our definition will use the push-forward of a finite cycle.

Let $p : T \to S$ be any morphism. If W is a irreducible closed subset of T finite along p, the image $V = f(W)$ is a closed irreducible subset of S and $d = [k(W) : k(V)]$ is finite. In this case we define the **push-forward** of the cycle W along p to be the cycle $p_* W = d \cdot V$; see [Ful84]. By additivity we may define the push-forward of any cycle which is finite along p.

Lemma 1.4. *Suppose that $f : T \to T'$ is a morphism of separated schemes of finite type over a Noetherian base S. Let W be an irreducible closed subset of T which is finite over S. Then $f(W)$ is closed and irreducible in T' and finite over S. If W is finite and surjective over S, then so is $f(W)$.*

Proof. By Ex.II.4.4 of [Har77], $f(W)$ is closed in T' and proper over S. Since $f(W)$ has finite fibers over S, it is finite over S by [EGA3, 4.4.2]. If $W \to S$ is surjective, so is $f(W) \to S$. $\square$

Given elementary correspondences $V \in Cor(X, Y)$ and $W \in Cor(Y, Z)$, form the intersection $T = (V \times Z) \cap (X \times W)$ in $X \times Y \times Z$. The composition $W \circ V$ of V and W is defined to be the push-forward of the finite correspondence $[T]$, along the projection $p : X \times Y \times Z \to X \times Z$; see [Ful84]. By lemma 1.7 below, the cycle $[T]$ is finite over $X \times Z$. Thus the push-forward $p_*[T]$ is defined; it is a finite correspondence from X to Z by lemma 1.4.

We can easily check that id_X is the identity of $Cor(X, X)$, and that the composition of finite correspondences is associative and bilinear (see [Man68] and [Ful84, 16.1]).

Definition 1.5. Let Cor_k be the category whose objects are the smooth separated schemes of finite type over k and whose morphisms from X to Y are elements of $Cor(X, Y)$. It follows from the above remarks that Cor_k is an additive category with $\emptyset$ as the zero object.

Lemma 1.6. *Let Z be an integral scheme, finite and surjective over a normal scheme S. Then for every morphism $T \to S$ with T connected, every component of $T \times_S Z$ is finite and surjective over T.*

Proof. See [EGA4, 14.4.4]. $\square$

Recall that two irreducible closed subsets Z_1 and Z_2 of a smooth scheme are said to intersect **properly** if $Z_1 \cap Z_2 = \emptyset$ or $\operatorname{codim}(Z_1 \cap Z_2) = \operatorname{codim} Z_1 + \operatorname{codim} Z_2$.

Lemma 1.7. *Let $V \subset X \times Y$ and $W \subset Y \times Z$ be irreducible closed subsets which are finite and surjective over X and Y respectively. Then $V \times Z$ and $X \times W$ intersect properly, and each component of the push-forward of the cycle $[T]$ of $T = (V \times Z) \cap (X \times W)$ is finite and surjective over X.*

Proof. Let $\tilde{V}$ and $\tilde{W}$ be the underlying integral subschemes associated to V and W respectively. Without loss of generality we can suppose both X and Y connected. We form the pullback of $\tilde{V}$ and $\tilde{W}$.

$$\begin{tikzcd}
\tilde{V} \times_Y \tilde{W} \arrow[r] \arrow[d] & \tilde{W} \arrow[r] \arrow[d, "\text{f.surj.}"] & Z \\
\tilde{V} \arrow[r] \arrow[d, "\text{f.surj.}"'] & Y & \\
X & &
\end{tikzcd}$$

By 1.6, each component of $\tilde{V} \times_Y \tilde{W}$ is finite and surjective over $\tilde{V}$ and therefore over X too. The image T of the evident map $\tilde{V} \times_Y \tilde{W} \to X \times Y \times Z$ is the intersection of $\tilde{V} \times Z$ and $\tilde{X} \times W$. Thus each irreducible component T_i of T is the image of an irreducible component of $\tilde{V} \times_Y \tilde{W}$. By 1.4, we know that each T_i is finite and surjective over X. Therefore $\dim T_i = \dim X$ for all i, i.e., $\tilde{V} \times Z$ and $X \times \tilde{W}$ intersect properly.

Let $p(T_i)$ denote the image of T_i under the map $p : X \times Y \times Z \to X \times Z$. By lemma 1.4, each $p(T_i)$ is an irreducible closed subscheme of $X \times Z$ which is finite and surjective over X. Since the components of $p_*[T]$ are the supports of the $p(T_i)$, we are done. □

Remark 1.8. It is possible to extend the definition of finite correspondences to correspondences between singular schemes. This uses the category Cor_S, where S is a Noetherian scheme; see [RelCy]. Since we will use only smooth schemes in these lectures, we describe this more general definition in the appendix of this lecture.

The additive category Cor_k is closely related to the category Sm/k of smooth schemes over k. Indeed, these categories have the same objects, and it is a routine computation (exercise!) to check that $\Gamma_g \circ \Gamma_f$ equals $\Gamma_{g \circ f}$. That is, there is a faithful functor $Sm/k \to Cor_k$, defined by:

$$X \mapsto X \qquad (f : X \to Y) \mapsto \Gamma_f.$$

The tensor product is another important feature of the category Cor_k.

Definition 1.9. If X and Y are two objects in Cor_k, their **tensor product** $X \otimes Y$ is defined to be the product of the underlying schemes over k:

$$X \otimes Y = X \times Y.$$

If V and W are elementary correspondences from X to X' and from Y to Y', then the cycle associated to the subscheme $V \times W$ by 1.3 gives a finite correspondence from $X \otimes Y$ to $X' \otimes Y'$.

It is easy to verify that $\otimes$ makes Cor_k a symmetric monoidal category (see [Mac71]).

Exercise 1.10. If $S = \operatorname{Spec} k$ then $Cor_k(S, X)$ is the group of zero-cycles in X. If W is a finite correspondence from $\mathbb{A}^1$ to X, and $s, t : \operatorname{Spec} k \to \mathbb{A}^1$ are k-points, show that the zero-cycles $W \circ \Gamma_s$ and $W \circ \Gamma_t$ are rationally equivalent (Cf. [Ful84, 1.6]).

Exercise 1.11. Let x be a closed point on X, considered as a correspondence from $S = \operatorname{Spec}(k)$ to X. Show that the composition $S \to X \to S$ is multiplication by the degree $[k(x) : k]$, and that $X \to S \to X$ is given by $X \times x \subset X \times X$.

Let L/k be a finite Galois extension with Galois group G and $T = \operatorname{Spec}(L)$. Prove that $Cor_k(T, T) \cong \mathbb{Z}[G]$ and that $T \to S \to T$ is $\sum_{g \in G} g \in \mathbb{Z}[G]$. Conclude that $Cor_k(S, Y) \cong Cor_k(T, Y)^G$ for every Y in Sm/k.

Exercise 1.12. If $k \subset F$ is a field extension, there is an additive functor $Cor_k \to Cor_F$ sending X to X_F. If F is finite and separable over k, there is an additive functor $Cor_F \to Cor_k$ sending U to U. These are adjoint: if U is smooth over F and X is smooth over k, there is a canonical identification:

$$Cor_F(U, X_F) = Cor_k(U, X).$$

Exercise 1.13. (a) Let F be a field extension of k and X and Y two smooth schemes over k. Writing X_F for $X \times_{\operatorname{Spec} k} \operatorname{Spec} F$ and so on, show that $Cor_F(X_F, Y_F)$ is the limit of the $Cor_E(X_E, Y_E)$ as E ranges over all finitely generated field extensions of k contained in F.

(b) Let $X \to S \to \operatorname{Spec}(k)$ be smooth morphisms, with S connected, and let F denote the function field of S. For every smooth scheme Y over k, show that $Cor_F(X \times_S \operatorname{Spec} F, Y \times_k \operatorname{Spec} F)$ is the direct limit of the $Cor_k(X \times_S U, Y)$

as U ranges over all non-empty open subschemes of S. In the special case $X = S$, this shows that $Cor_F(\operatorname{Spec} F, Y \times_k \operatorname{Spec} F) = \varinjlim Cor_k(U, Y)$.

(c) Show that (a) and (b) remain valid if Y is any scheme over k, using the definition 1.1 of $Cor_k(X, Y)$.

Appendix 1A - The category Cor_S

It is possible to generalize the notion of finite correspondence to construct a category Cor_S, associated to any Noetherian scheme S; see [RelCy]. The objects of this category are the schemes of finite type over S and the morphisms are the elements of an abelian group $Cor_S(X, Y)$ whose elements are the "universally integral" cycles $W \subset X \times_S Y$, each component of which is finite and surjective over X. The pull-backs of universally integral cycles are always defined.

In order to compose an elementary correspondence V in $Cor_S(X, Y)$ with a correspondence W in $Cor_S(Y, Z)$ we must form the pullback W along $V \to Y$ to a cycle in W_V in $V \times_S Z \subset X \times_S Y \times_S Z$ (See 1A.9). This is why we need to restrict to universally integral cycles, because not every cycle has such a pullback.

Relabeling, we are reduced to the following basic setup for pulling back cycles. We are given a cycle W in X, a structure map $X \to S$ and a map $V \to S$. The problem is to define a pullback cycle W_V in $X \times_S V$ in a natural way. This is easy if V is flat over S (see [Ful84, 1.7]), but in general the problem is quite difficult even for $V = \operatorname{Spec} K$.

One way to attack the problem is to use discrete valuation rings (DVR's), introducing the notion of pullback along a fat point of S. This approach was introduced in [RelCy]. Recall that if K is a field, a K-point of S (or point) is a morphism $\operatorname{Spec} K \to S$.

Definition 1A.1. A **fat point of** S is a DVR D, a field K and morphisms

$$\operatorname{Spec} K \to \operatorname{Spec} D \to S,$$

so that the point of $\operatorname{Spec} K$ goes to the closed point of $\operatorname{Spec} D$ and the generic point of $\operatorname{Spec} D$ goes to a generic point of S. We say that the fat point lies over the underlying K-point $\operatorname{Spec} K \to S$.

Every point s in S has a fat point lying over it in the sense that there is a field extension $k(s) \subset K$ and a fat point over $\operatorname{Spec} K \to S$. If s is a generic point of S, this is trivial ($D = K$). Otherwise, it is well known that for every two points $s < s'$ in a Noetherian scheme S there exists a DVR D and a map $\operatorname{Spec} D \to S$ sending the two points of $\operatorname{Spec} D$ to s and s'; see [EGA1, 6.5.8].

Theorem 1A.2. *Let D be a DVR with field of fractions F. If X is a scheme of finite type over D and W_F is closed in the generic fiber X_F then there exists a unique closed subscheme W_D of W_F in X which is flat over* $\operatorname{Spec} D$.

Proof. Locally X has coordinate ring A, X_F has coordinate ring $A \otimes_D F$, and W_F has coordinate ring $(A \otimes_D F)/(f_1, \ldots, f_n)$, where $f_i \in A$ for every $i = 1, \ldots, n$. Let R_0 be $A/(f_1, \ldots, f_n)$ and let R be R_0/I where I is the torsion submodule of the D-module R_0. Is is easy to see that R is independent of the choice of the f_i's. Locally W_D is $\operatorname{Spec} R$. □

We are not yet able to define the pullback along a K-point, but using the previous theorem we can define a pullback along a fat point.

Given a fat point of S over a K-point s, and a closed subscheme W in X, we may form the flat pullback W_F along $\operatorname{Spec} F \to S$ and the closed subscheme W_D as in 1A.2. Then we define the pullback of W to be the cycle associated to the fiber W_s of the scheme W_D over the closed point $\operatorname{Spec} K$ of D. The pullback $[W_s]$ is a cycle in $X_K = X \times_S \operatorname{Spec} K$. Thus for every fat point over s we have a candidate for the pullback of W along s. However, two fat points over the same K-point may give two possibly distinct candidates.

Example 1A.3. Let S be the node over a field k and X its normalization. There are two fat points over the singular point $s \in S$, corresponding to the two k-points of $X_s = \{p_0, p_1\}$. The pullbacks of $W = X$ along these fat points are $[p_0]$ and $[p_1]$, respectively.

If W is flat and equidimensional over S (or s is generic) the pullback just defined coincides with the classic pullback of a cycle along the K-point (see [RelCy, 3.2.4]), so it is independent of the choice of the fat point.

In order to have a useful object we need to get rid of the dependency of the pullback from the choice of the fat point.

Definition 1A.4. Let $W = \sum n_i W_i$ be a cycle on X. We say that W is *dominant* over S if each term W_i of W is dominant over a component of S. We say that a dominant cycle W is a **relative cycle** on X over S if its pullbacks coincide along all fat points over any common K-point; see [RelCy, 3.1.3].

As in [RelCy], we write $Cycl(X/S, r)$ for the free abelian group of the relative cycles W on X over S such that each component has dimension r over S. It turns out that every effective relative cycle in $Cycl(X/S, r)$ must be equidimensional over S; see [RelCy, 3.1.7]. If S is normal, this is also a sufficient condition; the following result is proven in [RelCy, 3.4.2].

Theorem 1A.5. *If S is normal or geometrically unibranch and W is a cycle on X which is dominant equidimensional over S, then W is a relative cycle.*

We now have a good definition for the pullback of a relative cycle along a map $\operatorname{Spec} K \to S$ supporting a fat point. We want to generalize this to pullbacks along any Zariski point s of S. However, there may be no fat points over s. For example, it may be that every fat point $\operatorname{Spec} K \to S$ has K inseparable over $k(s)$. To fix this, it turns out that we need to invert the characteristic p.

Example 1A.6. Let k be a purely inseparable extension of k_0 with $[k : k_0] = p$ and set $X = \operatorname{Spec}(k[t])$. Let $S = \operatorname{Spec} A$ where $A \subset k[t]$ is the ring of polynomials $f(t)$ where $f(0) \in k_0$. If $T_0 = \operatorname{Spec} k_0$ is the origin of S then the pullback of $W = X$ to $T = X \times_S T_0 = \operatorname{Spec} k$ is $1/p$ times $[T]$, because every fat point of S must lie over a field extension of k.

Given $f : V \to S$, the pullback W_V of a relative cycle W is a unique and well-defined relative cycle of $X \times_S V$ over V, except that the coefficients may lie in $\mathbb{Z}[1/p]$ in characteristic p. It is characterized by the fact that for every point v of V and every fat point of V over v, the pullbacks of $(W_V)_v$ and $W_{f(v)}$ agree. See [RelCy, 3.3.1].

See Example 3.5.10 in [RelCy] for a relative cycle for which the coefficient $1/p$ occurs in its pullbacks, yet both X and S are normal.

Definition 1A.7. A relative cycle W is called **universally integral** when its pullbacks W_V always have integer coefficients; see [RelCy, 3.3.9].

We define $c(X/S, 0)$ to be the free abelian group on the universally integral relative cycles of X which are finite and surjective over S. Finally we set $Cor_S(X,Y) = c(X \times_S Y/X, 0)$. That is, $Cor_S(X,Y)$ is the group of universally integral cycles on $X \times_S Y$ whose support is finite over X (i.e., proper over X of relative dimension 0).

In [RelCy] the notation $z(X/S,0)$ was used for the subgroup of $Cycl(X/S,0)$ generated by universally integral cycles, and the notation $c(X/S,0)$ was introduced for the subgroup generated by the proper cycles in $z(X/S,0)$.

The following theorem was proved in [RelCy, 3.3.15] and [RelCy, 3.4.8].

Theorem 1A.8. *Any relative cycle of X over S is universally integral provided that either*

1. *S is regular, or*

2. *X is a smooth curve over S.*

Definition 1A.9. The composition of relative cycles $V \in Cor_S(X,Y)$ and $W \in Cor_S(Y,Z)$ is defined as follows. Form the pullback W_V of W with respect to the map $V \to Y$. The composition $W \circ V$ is defined to be the push-forward of W_V along the projection $p : X \times Y \times Z \to X \times Z$. By [RelCy, 3.7.5], the composition will be a universally integral cycle which is finite and surjective over X.

Example 1A.10. By definition, $c(X/S,0) = Cor_S(S,X)$. If S and X are smooth over a field k, then clearly $Cor_S(S,X) \subseteq Cor_k(S,X)$ via the embedding of X in $S \times X$. Hence, for every map $S' \to S$, there is a map $c(X/S,0) \to c(X \times_S S'/S',0)$ induced by composition in Cor_k.

$$\begin{tikzcd}
c(X/S,0) \arrow[r, hook] \arrow[d, dotted] & Cor_k(S,X) \arrow[d] \\
c(X \times_S S'/S',0) \arrow[r, hook] & Cor_k(S',X)
\end{tikzcd}$$

Example 1A.11. If $S = \operatorname{Spec} k$ for a field k and X and Y are smooth over S, then the group $Cor_S(X,Y) = c(X \times Y/X,0)$ agrees with the group $Cor_k(X,Y)$ of definition 1.1.

To see this, note that $c(X \times Y/X,0) \subseteq Cor_k(X,Y)$ by definition. By 1A.5 and 1A.8, every cycle in $X \times Y$ which is finite and surjective over X is a universally integral relative cycle, so we have equality.

Since composition in Cor_S (as defined in 1A.9) evidently agrees with composition in Cor_k, we see that Cor_k is just the restriction of Cor_S to Sm/k.

Example 1A.12. Suppose that $V \subset S$ is a closed immersion of regular schemes and let W be an equidimensional cycle on a scheme X of finite type over S. It it shown in [RelCy, 3.5.8] that the pullback cycle W_V coincides with the image of W under the pull-back homomorphism for the map $V \times_S X \to X$ as defined in [Ser65] and [Ful84], using an alternating sum of Tor terms.

Lecture 2

Presheaves with transfers

In order to define motivic cohomology we need to introduce the notion of a presheaf with transfers. In this lecture we develop the basic properties of presheaves with transfers.

Definition 2.1. A **presheaf with transfers** is a contravariant additive functor $F : Cor_k \to \mathbf{Ab}$. We will write $PreSh(Cor_k)$, or $\mathbf{PST}(k)$ or $\mathbf{PST}$ if the field is understood, for the functor category whose objects are the presheaves with transfers and whose morphisms are natural transformations.

By additivity, there is a pairing $Cor_k(X, Y) \otimes F(Y) \to F(X)$ for all F, X and Y.

Restricting to the subcategory Sm/k of Cor_k, we see that a presheaf with transfers F may be regarded as a presheaf of abelian groups on Sm/k which is equipped with extra "transfer" maps $F(Y) \to F(X)$ indexed by the finite correspondences from X to Y.

Example 2.2. Every constant presheaf A on Sm/k may be regarded as a presheaf with transfers. If W is an elementary correspondence from X to Y (both connected), the homomorphism $A \to A$ defined by W is multiplication by the degree of W over X.

The following theorem is a special case of a well known result on functor categories, see [Wei94] 1.6.4 and Exercises 2.3.7 and 2.3.8.

Theorem 2.3. *The category* $\mathbf{PST}(k)$ *is abelian and has enough injectives and projectives.*

Example 2.4. The sheaf $\mathcal{O}^*$ of global units and the sheaf $\mathcal{O}$ of global functions are two examples of a presheaves with transfers.

Recall first that if X is normal and $W \to X$ is finite and surjective then there is a norm map $N : \mathcal{O}^*(W) \to \mathcal{O}^*(X)$ induced from the usual norm map on the function fields, $k(W)^* \to k(X)^*$. Indeed if $f \in \mathcal{O}^*(W)$ then Nf and Nf^{-1} are both in the integrally closed subring $\mathcal{O}(X)$ of $k(X)$.

Similarly, there is a trace map $\mathrm{Tr} : \mathcal{O}(W) \to \mathcal{O}(X)$ induced from the usual trace map on the function fields, $k(W) \to k(X)$. Indeed if $f \in \mathcal{O}(W)$ then $\mathrm{Tr}f$ belongs to the integrally closed subring $\mathcal{O}(X)$ of $k(X)$.

If $W \subset X \times Y$ is an elementary correspondence from X to Y, we define the transfer map $\mathcal{O}^*(Y) \to \mathcal{O}^*(X)$ associated to W to be the composition:

$$\mathcal{O}^*(Y) \longrightarrow \mathcal{O}^*(W) \xrightarrow{N} \mathcal{O}^*(X).$$

We define the transfer $\mathcal{O}(Y) \to \mathcal{O}(X)$ associated to W to be the composition

$$\mathcal{O}(Y) \longrightarrow \mathcal{O}(W) \xrightarrow{\mathrm{Tr}} \mathcal{O}(X).$$

We omit the verification that these transfers are compatible with the composition in Cor_k. It is clear from the transfer formula that the subsheaf μ_n of n^{th} roots of unity in $\mathcal{O}^*$ is also a presheaf with transfers, and that the subsheaf k of $\mathcal{O}$ is just the constant sheaf with transfers described in 2.2.

Example 2.5. The classical Chow groups $CH^i(-)$ are presheaves with transfers. To see this, we need to construct a map $\phi_W : CH^i(Y) \to CH^i(X)$ for each elementary finite correspondence W from a smooth scheme X to a smooth scheme Y, and check that this defines a functor from Cor_k to abelian groups.

The correspondence homomorphism ϕ_W is given by the formula $\phi_W(\alpha) = q_*(W \cdot p^*\alpha)$, where $\alpha \in CH^i(Y)$. Here $p^* : CH^i(Y) \to CH^i(X \times Y)$ is the flat pullback along the projection $X \times Y \to Y$, the '$\cdot$' is the intersection product (see 17A.1), and $q : X \times Y \to X$ is the projection. If Y were proper, this would be exactly the formula given in Chapter 16 of [Ful84]. For general Y, we need to observe that $W \cdot p^*\alpha$ has finite support over X, so that the push-forward $q_*(W \cdot p^*\alpha)$ is defined in $CH^i(X)$.

The verification that the definition of ϕ_W is compatible with the composition of correspondences is now a routine calculation using the projection formula; it is practically the same as the calculation in the proper case, which is given in [Ful84, 16.1.2].

Example 2.6. The functor K_0, considered as a presheaf of abelian groups on Sm/k, has no extension to a presheaf with transfers. To see this, it suffices to find a finite étale cover $f : Y \to X$ of degree 2 and an element $x \in K_0(X)$ such that $f^*(x) = 0$ but $2x \neq 0$. Indeed, if $\Phi \in Cor(X, Y)$ is the canonical "transfer" morphism defined by f, then $f \circ \Phi = 2$ in $Cor(X, X)$ (cf. 1.11), so any presheaf with transfers F would have $F(\Phi)f^*(x) = 2x$ for all $x \in F(X)$.

Let $\mathcal{L}$ be a line bundle on a smooth variety X satisfying $\mathcal{L}^2 \cong \mathcal{O}_X$ but $[\mathcal{L} \oplus \mathcal{L}] \neq [\mathcal{O}_X \oplus \mathcal{O}_X]$ in $K_0(X)$. It is well-known that such $\mathcal{L}$ exists; see [Swa62]. It is also well-known that there is an étale cover $f : Y \to X$ of degree 2 with $Y = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L})$; see [Har77, IV Ex.2.7]. Since $f^*\mathcal{L} \cong \mathcal{O}_Y$, the element $x = [\mathcal{L}] - [\mathcal{O}_X]$ of $K_0(X)$ satisfies $f^*(x) = 0$ but $2x \neq 0$, as required.

Representable functors provide another important class of presheaves with transfers. We will use the notation $\mathbb{Z}_{tr}(X)$, which was introduced in [SV00]; the alternate terminology $L(X)$ was used in [TriCa], while $c_{equi}(X/\operatorname{Spec} k, 0)$ was used in [RelCy] and [CohTh].

By the Yoneda lemma, representable functors provide embeddings of Sm/k and Cor_k into an abelian category, namely $\mathbf{PST}(k)$:

$$\begin{array}{ccccc} Sm/k & \longrightarrow & Cor_k & \longrightarrow & \mathbf{PST}(k). \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}_{tr}(X) \end{array}$$

Definition 2.7. If X is a smooth scheme over k we let $\mathbb{Z}_{tr}(X)$ denote the presheaf with transfers represented by X, so that $\mathbb{Z}_{tr}(X)(U) = Cor(U, X)$. By the Yoneda lemma,

$$Hom_{\mathbf{PST}}(\mathbb{Z}_{tr}(X), F) \cong F(X).$$

It follows that $\mathbb{Z}_{tr}(X)$ is a projective object in $\mathbf{PST}(k)$.

For every X and U, $\mathbb{Z}_{tr}(X)(U)$ is the group of finite correspondences from U to X and the map $\mathbb{Z}_{tr}(X)(U) \to \mathbb{Z}_{tr}(X)(V)$ associated to a morphism $f : V \to U$ is defined to be composition with the correspondence associated to f.

We will write $\mathbb{Z}$ for the presheaf with transfers $\mathbb{Z}_{tr}(\operatorname{Spec} k)$; it is just the constant Zariski sheaf $\mathbb{Z}$ on Sm/k, equipped with the transfer maps of 2.2. Thus the structure map $X \to \operatorname{Spec} k$ induces a natural map $\mathbb{Z}_{tr}(X) \to \mathbb{Z}$.

Here are three exercises. Carefully writing up their solutions requires some knowledge about cycles, such as that found in [Ful84].

Exercise 2.8. If F is a presheaf with transfers and T is a smooth scheme, define $F^T(U) = F(U \times T)$. Show that F^T is a presheaf with transfers and that every morphism $S \to T$ induces a morphism $F^T \to F^S$ of presheaves with transfers. If F is constant and T is geometrically connected, then $F^T = F$.

Exercise 2.9. If $k \subset L$ is a separable field extension, every X in Sm/L is an inverse limit of schemes X_α in Sm/k. For every presheaf with transfers F over k, we set $F(X) = \varinjlim F(X_\alpha)$. Show that this makes F a presheaf with transfers over L.

Exercise 2.10. Let X be a (non-smooth) scheme of finite type over k. For each smooth U, define $\mathbb{Z}_{tr}(X)(U)$ to be the group $Cor(U, X)$ of 1.1. Show that the composition $\circ$ defined after 1.4 makes $\mathbb{Z}_{tr}(X)$ into a presheaf with transfers.

Given a pointed scheme (X, x), we define $\mathbb{Z}_{tr}(X, x)$ to be the cokernel of the map $x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)$ associated to the point $x : \operatorname{Spec} k \to X$. Since x_* splits the structure map $\mathbb{Z}_{tr}(X) \to \mathbb{Z}$, we have a natural splitting $\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$.

Of particular interest to us are the pointed scheme $\mathbb{G}_m = (\mathbb{A}^1 - \{0\}, 1)$, and the presheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m) = \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}, 1)$.

Definition 2.11. If (X_i, x_i) are pointed schemes for $i = 1, \dots, n$ we define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$, or $\mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_n)$, to be:

$$\operatorname{coker}\left(\bigoplus_i \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}_i \cdots \times X_n) \xrightarrow{id \times \cdots \times x_i \times \cdots \times id} \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \right).$$

By definition $\mathbb{Z}_{tr}((X, x)^{\wedge 1}) = \mathbb{Z}_{tr}(X, x)$ and $\mathbb{Z}_{tr}((X, x)^{\wedge q}) = \mathbb{Z}_{tr}((X, x) \wedge \ldots \wedge (X, x))$ for $q > 0$. By convention $\mathbb{Z}_{tr}((X, x)^{\wedge 0}) = \mathbb{Z}$ and $\mathbb{Z}_{tr}((X, x)^{\wedge q}) = 0$ when $q < 0$.

Lemma 2.12. *The presheaf $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. In particular, it is a projective object of* **PST**.

Proof. This is a consequence of the stronger fact that the following sequence of presheaves with transfers is split exact (see [Wei94, 1.4.1]):

$$0 \to \mathbb{Z} \overset{\{x_i\}}{\to} \oplus_i \mathbb{Z}_{tr}(X_i) \to \oplus_{i,j} \mathbb{Z}_{tr}(X_i \times X_j) \to \ldots$$
$$\ldots \to \oplus_{i,j} \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}^i \cdots \hat{X}^j \cdots \times X_n) \to \oplus_i \mathbb{Z}_{tr}(X \times \cdots \hat{X}^i \cdots \times X_n) \to$$
$$\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) \to 0.$$

Since $\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n)$ is a summand of a projective object, it is projective. $\square$

To illustrate this lemma note $\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$ and that:

$$\mathbb{Z}_{tr}(X_1 \times X_2) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X_1, x_1) \oplus \mathbb{Z}_{tr}(X_2, x_2) \oplus \mathbb{Z}_{tr}(X_1 \wedge X_2).$$

We shall also need a functorial construction of a chain complex associated to a presheaf with transfers. For this we use the cosimplicial scheme $\Delta^\bullet$ over k which is defined by:

$$\Delta^n = \operatorname{Spec} k[x_0, \ldots, x_n] / \left(\sum_{i=0}^{n} x_i = 1 \right).$$

The j^{th} face map $\partial_j : \Delta^n \to \Delta^{n+1}$ is given by the equation $x_j = 0$. Although this construction is clearly taken from topology, the use of $\Delta^\bullet$ in an algebraic setting originated with D. Rector in [Rec71].

Definition 2.13. If F is a presheaf of abelian groups on Sm/k, $F(\Delta^\bullet)$ and $F(U \times \Delta^\bullet)$ are simplicial abelian groups. We will write $C_\bullet F$ for the simplicial presheaf $U \mapsto F(U \times \Delta^\bullet)$, i.e., $C_n(F)(U) = F(U \times \Delta^n)$. If F is a presheaf with transfers, $C_\bullet F$ is a simplicial presheaf with transfers by 2.8.

As usual, we can take the alternating sum of the face maps to get a chain complex of presheaves (resp., presheaves with transfers) which (using $*$ in place of $\bullet$), we will call $C_* F$. It sends U to the complex of abelian groups:

$$\ldots \to F(U \times \Delta^2) \to F(U \times \Delta^1) \to F(U) \to 0.$$

Both $F \mapsto C_\bullet F$ and $F \mapsto C_* F$ are exact functors. Moreover, the Dold-Kan correspondence (see [Wei94, 8.4.1]), which describes an equivalence between simplicial objects and positive chain complexes, associates to $C_\bullet F$ a normalized subcomplex $C_*^{DK} F$ of the complex $C_* F$, which is quasi-isomorphic to the complex $C_* F$.

If A is the constant presheaf with transfers $A(U) = A$ then $C_* A$ is the complex $\cdots \to A \xrightarrow{id} A \xrightarrow{0} A \to 0$; it is quasi-isomorphic to $C_*^{DK}(A)$, which is A regarded as a complex concentrated in degree zero.

Definition 2.14. A presheaf F is **homotopy invariant** if for every X the map $p^* : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism. As $p : X \times \mathbb{A}^1 \to X$ has a section, p^* is always split injective. Thus homotopy invariance of F is equivalent to p^* being onto.

The homotopy invariant presheaves of abelian groups form a Serre subcategory of presheaves, meaning that if $0 \to F_0 \to F_1 \to F_2 \to 0$ is an exact sequence of presheaves then F_1 is homotopy invariant if and only if both F_0 and F_2 are. In particular if F and G are homotopy invariant presheaves with transfers then the kernel and the cokernel of every map $f : F \to G$ are homotopy invariant presheaves with transfers.

Let $i_\alpha : X \hookrightarrow X \times \mathbb{A}^1$ be the inclusion $x \mapsto (x, \alpha)$. We shall write i_α^* for $F(i_\alpha) : F(X \times \mathbb{A}^1) \to F(X)$.

Lemma 2.15. *F is homotopy invariant if and only if*

$$i_0^* = i_1^* : F(X \times \mathbb{A}^1) \to F(X) \qquad \textit{for all } X.$$

Proof. ([Swa72, 4.1]) One direction is clear, so suppose that $i_0^* = i_1^*$ for all X. Applying F to the multiplication map $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, $(x, y) \mapsto xy$, yields the diagram

$$\begin{tikzcd}
 & F(X \times \mathbb{A}^1) \arrow[dl, "1_{X \times \mathbb{A}^1}"'] \arrow[r, "i_0^*"] \arrow[d, "(1_X \times m)^*"] & F(X) \arrow[d, "p^*"] \\
F(X \times \mathbb{A}^1) & F(X \times \mathbb{A}^1 \times \mathbb{A}^1) \arrow[l, "(i_1 \times 1_{\mathbb{A}^1})^*"'] \arrow[r, "(i_0 \times 1_{\mathbb{A}^1})^*"] & F(X \times \mathbb{A}^1).
\end{tikzcd}$$

Hence $p^* i_0^* = (1 \times i_0)^* m^* = (1 \times i_1)^* m^* = id$. Since $i_0^* p^* = id$, p^* is an isomorphism. $\square$

Definition 2.16. For $i = 0, \dots, n$ we define $\theta_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ to be the map that sends the vertex v_j to $v_j \times \{0\}$ for $j \le i$ and to $v_{j-1} \times \{1\}$ otherwise. (See Figure 2.1.) These are the algebraic analogues of the top dimensional simplices in the standard simplicial decomposition of the polyhedron $\Delta^n \times \Delta^1$.

Lemma 2.17. *Let F be a presheaf. Then the maps $i_0^*, i_1^* : C_*F(X \times \mathbb{A}^1) \to C_*F(X)$ are chain homotopic.*

Proof. The maps θ_i defined in 2.16 induce maps

$$h_i = F(1_X \times \theta_i) : C_n F(X \times \mathbb{A}^1) \to C_{n+1} F(X).$$

The h_i form a simplicial homotopy ([Wei94, 8.3.11]) from $i_1^* = \partial_0 h_0$ to $i_0^* = \partial_{n+1} h_n$. By [Wei94, 8.3.13], the alternating sum $s_n = \sum (-1)^i h_i$ is a chain homotopy from i_1^* to i_0^*. $\square$

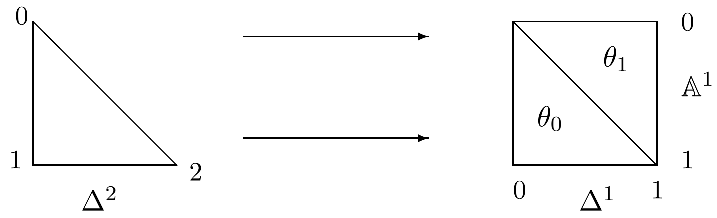

Figure 2.1: Simplicial decomposition of $\Delta^n \times \mathbb{A}^1$

Combining 2.15 and 2.17, we obtain

Corollary 2.18. *If F is a presheaf then the homology presheaves*

$$H_n C_* F : X \mapsto H_n C_* F(X)$$

are homotopy invariant for all n.

Example 2.19. ([Swa72, 4.2]) The surjection $F \to H_0 C_* F$ is the universal morphism from F to a homotopy invariant presheaf.

Exercise 2.20. Set $H_0^{sing}(X/k) = H_0 C_* \mathbb{Z}_{tr}(X)(\operatorname{Spec} k)$. Show that there is a natural surjection from $H_0^{sing}(X/k)$ to $CH_0(X)$, the Chow group of zero cycles modulo rational equivalence (see exercise 1.10). If X is projective, $H_0^{sing}(X/k) \cong CH_0(X)$. If $X = \mathbb{A}^1$, show that $H_0^{sing}(\mathbb{A}^1/k) = \mathbb{Z}$. We will return to this point in 7.1.

Lemma 2.21. *Let F be a presheaf of abelian groups. Suppose that for every smooth scheme X there is a natural homomorphism $h_X : F(X) \to F(X \times \mathbb{A}^1)$ which fits into the diagram*

$$\begin{tikzcd}
 & F(X) \arrow[dl, "0"'] \arrow[d, "h_X"] \arrow[dr, "id"] & \\
F(X) & F(X \times \mathbb{A}^1) \arrow[l, "F(i_0)"] \arrow[r, "F(i_1)"'] & F(X)
\end{tikzcd}$$

Then the complex $C_* F$ is chain contractible.

The assertion that h_X is natural means that for every map $f : X \to Y$

we have a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "h_X"] & F(X \times \mathbb{A}^1) \\ F(Y) \arrow[u] \arrow[r, "h_Y"] & F(Y \times \mathbb{A}^1) \arrow[u] \end{tikzcd}$$

Proof. By naturality, h_X induces a map $C_*h : C_*F(X) \to C_*F(X \times \mathbb{A}^1)$. By 2.17, the identity map $id = i_1^*(C_*h)$ is chain homotopic to $0 = i_0^*(C_*h)$. ☐

Example 2.22. The prototype for lemma 2.21 is the sheaf of global functions. The complex $C_*\mathcal{O}$ is chain contractible, because $\mathcal{O}(X \times \mathbb{A}^1) \cong \mathcal{O}(X)[t]$ and $h_X(f) = tf$ satisfies the conditions of 2.21.

Here is a second application of 2.21. Note that the projection $p : X \times \mathbb{A}^1 \to X$ induces a map $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$.

Corollary 2.23. $C_*\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to C_*\mathbb{Z}_{tr}(X)$ *is a chain homotopy equivalence.*

Proof. Let F denote the cokernel of $\mathbb{Z}_{tr}(i_0) : \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(X \times \mathbb{A}^1)$ induced by $i_0 : X \to X \times \mathbb{A}^1$. That is, each $F(U)$ is the cokernel of $Cor(U, X) \to Cor(U, X \times \mathbb{A}^1)$. Let H_U denote the composition of the product with $\mathbb{A}^1$ and multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$:

$$Cor(U, X \times \mathbb{A}^1) \to Cor(U \times \mathbb{A}^1, (X \times \mathbb{A}^1) \times \mathbb{A}^1) \to Cor(U \times \mathbb{A}^1, X \times \mathbb{A}^1).$$

Since H_U sends $Cor(U, X \times \{0\})$ to $Cor(U \times \mathbb{A}^1, X \times \{0\})$, it induces a natural map $h_U : F(U) \to F(U \times \mathbb{A}^1)$. For $U = X \times \mathbb{A}^1$ it is easy to see that the composition of H_U with $i_0, i_1 : U \to U \times \mathbb{A}^1$ sends $1_U \in Cor(U, X \times \mathbb{A}^1)$ to the projection $i_0 p : U \to X \to X \times \mathbb{A}^1$ and 1_U, respectively. Therefore $F(i_0)h_U(1_U) = 0$ and $F(i_1)h_U(1_U) = 1_U$ for $U = X \times \mathbb{A}^1$. For any other U, every element $\bar{f} \in F(U)$ is the image of $1_{X \times \mathbb{A}^1}$ under some correspondence $f : U \to X \times \mathbb{A}^1$, so again $F(i_0)h_U(\bar{f}) = 0$ and $F(i_1)h_U(\bar{f}) = \bar{f}$. Therefore 2.21 applies to show that C_*F is chain contractible. Since $C_*\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \cong C_*\mathbb{Z}_{tr}(X) \oplus C_*F$, we are done. ☐

An elementary $\mathbb{A}^1$-*homotopy* between two morphisms $f, g : X \to Y$ is a map $h : X \times \mathbb{A}^1 \to Y$ so that f and g are the restrictions of h along $X \times 0$ and $X \times 1$. This relation is not transitive (exercise!). To correct this, we pass to correspondences.

Definition 2.24. We say that two finite correspondences from X to Y are $\mathbb{A}^1$**-homotopic** if they are the restrictions along $X \times 0$ and $X \times 1$ of an element of $Cor(X \times \mathbb{A}^1, Y)$. This is an equivalence relation on $Cor(X, Y)$. The sum and composition of $\mathbb{A}^1$-homotopic maps are $\mathbb{A}^1$-homotopic, so the $\mathbb{A}^1$-homotopy classes of finite correspondences form the morphisms of an additive category.

We say that $f : X \to Y$ is an $\mathbb{A}^1$**-homotopy equivalence** if there exists a $g : Y \to X$ so that fg and gf are $\mathbb{A}^1$-homotopic to the identity.

The projection $p : X \times \mathbb{A}^1 \to X$ is the prototype of an $\mathbb{A}^1$-homotopy equivalence; its $\mathbb{A}^1$-homotopy inverse is given by the zero-section.

Lemma 2.25. *If $f : X \to Y$ is an $\mathbb{A}^1$-homotopy equivalence with $\mathbb{A}^1$-homotopy inverse g, then $f_* : C_*\mathbb{Z}_{tr}(X) \to C_*\mathbb{Z}_{tr}(Y)$ is a chain homotopy equivalence with chain homotopy inverse g_*.*

Proof. Applying $C_*\mathbb{Z}_{tr}$ to the data gives a diagram

$$\begin{tikzcd}
 & C_*\mathbb{Z}_{tr}(X) & \\
C_*\mathbb{Z}_{tr}(X) \arrow[ur, "g_*f_*"] \arrow[r, "(i_0)_*", "\cong"'] & C_*\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \arrow[u, "h_*"'] & C_*\mathbb{Z}_{tr}(X) \arrow[ul, "1_X"'] \arrow[l, "(i_1)_*"', "\cong"]
\end{tikzcd}$$

and similarly for Y. The horizontal maps are chain homotopy equivalences by 2.23, and are homotopy inverses to p_*. From the right triangle, $h_* \simeq p_*$. From the left triangle, we get $g_*f_* \simeq 1_X$. Similarly, the diagram for Y gives $f_*g_* \simeq 1_Y$. Hence $f_* : C_*\mathbb{Z}_{tr}(X) \to C_*\mathbb{Z}_{tr}(Y)$ is a chain homotopy equivalence with inverse g_*. $\square$

Exercise 2.26. Show that there is a natural identification for every X and Y:

$$H_0C_*\mathbb{Z}_{tr}(Y)(X) = Cor(X, Y)/\mathbb{A}^1\text{-homotopy}.$$

We will return to the subject of $\mathbb{A}^1$-homotopy in lectures 7, 9, 13, and 14; see 7.2, 9.8 and 14.11.

The motive associated to X will be the class $M(X)$ of $C_*\mathbb{Z}_{tr}(X)$ in an appropriate triangulated category $\mathbf{DM}^{\text{eff},-}_{Nis}(k, R)$ constructed in 14.1 from the derived category of $\mathbf{PST}(k)$. By 2.23, we have $M(X) \cong M(X \times \mathbb{A}^1)$ for all X. More generally, any $\mathbb{A}^1$-homotopy equivalence $X \to Y$ induces an isomorphism $M(X) \cong M(Y)$ by 2.25.

Exercise 2.27. If $k \subset F$ is a finite separable field extension, exercise 1.12 implies that there are adjoint functors $i^* : \mathbf{PST}(k) \to \mathbf{PST}(F)$, $i_* : \mathbf{PST}(F) \to \mathbf{PST}(k)$. Show that there is a natural transformation $\pi : i^*i_*M \to M$ whose composition $\pi\eta$ with the adjunction map $\eta : M \to i^*i_*M$ is multiplication by $[F : k]$ on M.

Lecture 3

Motivic cohomology

Using the tools developed in the last lecture, we will define motivic cohomology. It will be hypercohomology with coefficients in the special cochain complexes $\mathbb{Z}(q)$, called motivic complexes.

Definition 3.1. For every integer $q \geq 0$ the **motivic complex** $\mathbb{Z}(q)$ is defined as the following complex of presheaves with transfers:

$$\mathbb{Z}(q) = C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))[-q].$$

We consider $\mathbb{Z}(q)$ to be a bounded above cochain complex; the shifting convention for $[-q]$ implies that the terms $\mathbb{Z}(q)^i = C_{q-i}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$ vanish whenever $i > q$, and the term with $i = q$ is $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$.

If A is any other abelian group then $A(q) = \mathbb{Z}(q) \otimes A$ is another complex of presheaves with transfers.

When $q = 0$, we have $\mathbb{Z}(0) = C_*(\mathbb{Z})$. As observed after 2.13 above, $\mathbb{Z}(0)$ is quasi-isomorphic to $\mathbb{Z}$, regarded as a complex concentrated in degree 0.

When $q = 1$, we have $\mathbb{Z}(1) = C_*\mathbb{Z}_{tr}(\mathbb{G}_m)[-1]$. We will give another description of $\mathbb{Z}(1)$ in the next lecture.

By convention $\mathbb{Z}(q) = 0$ if $q < 0$.

We now show that these complexes of presheaves are actually complexes of sheaves with respect to the Zariski topology.

Lemma 3.2. *For every scheme Y over k, $\mathbb{Z}_{tr}(Y)$ is a sheaf in the Zariski topology, and $C_*\mathbb{Z}_{tr}(Y)$ is a chain complex of sheaves.*

Similarly, if A is any abelian group, the proof of 3.2 shows that $A\otimes\mathbb{Z}_{tr}(Y)$ is a sheaf in the Zariski topology, and $A\otimes C_*\mathbb{Z}_{tr}(Y)$ is a complex of sheaves.

Proof. We have to prove that whenever U is covered by U_1 and U_2 the sequence

$$0 \to Cor(U,Y) \xrightarrow{\text{diag}} Cor(U_1,Y)\oplus Cor(U_2,Y) \xrightarrow{(+,-)} Cor(U_1\cap U_2,Y)$$

is exact. We may suppose that U is connected and therefore (being smooth) irreducible. As every finite correspondence from U to Y is dominant over U, it is completely determined by the fiber at the generic point of U. Hence $Cor(U,Y)$ injects into each $Cor(U_i,Y)$.

To see that the sequence is exact at the other spot, take cycles $Z_1 = \sum_{i\in I} m_i Z_{1i} \subset U_1\times Y$ and $Z_2 = \sum_{j\in J} n_j Z_{2j} \subset U_2\times Y$ that coincide on $(U_1\cap U_2)\times Y$. It is possible to pair up the Z_{1i} and Z_{2j}, since they are determined by their fibers at the common generic point of U, U_1 and U_2. Hence there is a bijection between I and J such that, if $i\in I$ corresponds to $j\in J$ then $m_i=n_j$ and the restrictions of Z_{1i} and Z_{2j} agree in $(U_1\cap U_2)\times Y$. Thus we may assume that Z_1 and Z_2 are elementary correspondences. But then their union $Z=Z_1\cup Z_2$ in $U\times Y$ is a finite correspondence from U to Y, and its restriction to both $U_i\times Y$ is Z_i, i.e., Z is a preimage of the pair.

Now whenever F is a sheaf and X is smooth, each presheaf $U\mapsto F(U\times X)$ is also a sheaf for the Zariski topology. In particular each C_nF is a sheaf and C_*F is a complex of sheaves. Thus $C_*\mathbb{Z}_{tr}(Y)$ is a complex of Zariski sheaves. □

We have already seen (in exercises 2.20 and 2.26 above) that the complex $C_*\mathbb{Z}_{tr}(Y)$ is not exact. There we showed that the last map may not be surjective, because its cokernel $H_0C_*\mathbb{Z}_{tr}(Y)(S) = Cor(S,Y)/\mathbb{A}^1$-homotopy can be non-zero. When $S=\mathrm{Spec}(k)$, it is the group $H_0^{sing}(Y/k)$ described in exercise 2.20 above and 7.3 below.

Recall that the (small) Zariski site X_{zar} over a scheme X is the category of open subschemes of X, equipped with the Zariski topology.

Lemma 3.3. *The restriction $\mathbb{Z}(q)_X$ of $\mathbb{Z}(q)$ to the Zariski site over X is a complex of sheaves in the Zariski topology.*

Similarly, if A is any abelian group, $A(q)$ is a complex of Zariski sheaves.

Proof. Set $Y = (\mathbb{A}^1 - \{0\})^q$. By lemma 3.2 we know that $C_*(\mathbb{Z}_{tr}(Y))$ is a complex of sheaves. The complex $\mathbb{Z}(q)[q]$ is a direct summand of $C_*(\mathbb{Z}_{tr}(Y))$ by lemma 2.12, so it must be a complex of sheaves too. □

Note that $A(q)$ represents the derived sheaf tensor product $\mathbb{Z}(q) \otimes^{\mathbf{L}} A$, since $\mathbb{Z}(q)$ is a flat complex of sheaves.

Definition 3.4. The **motivic cohomology groups** $H^{p,q}(X, \mathbb{Z})$ are defined to be the hypercohomology of the motivic complexes $\mathbb{Z}(q)$ with respect to the Zariski topology:

$$H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)).$$

If A is any abelian group, we define:

$$H^{p,q}(X, A) = \mathbb{H}^p_{Zar}(X, A(q)).$$

Vanishing Theorem 3.5. *For every smooth scheme X and any abelian group A, we have $H^{p,q}(X, A) = 0$ when $p > q + \dim X$.*

Proof. By definition, the complex $\mathbb{Z}(q)$ is zero in degrees bigger then q. Since $H^i_{zar}(X, F)$ vanishes for every sheaf F when $i > \dim X$, the result is now an immediate consequence of the hypercohomology spectral sequence. □

We will prove in 19.3 that, for every smooth variety X and any abelian group A, we have $H^{p,q}(X, A) = 0$ for $p > 2q$ as well.

Remark 3.6. The groups $H^{p,q}(X, \mathbb{Z})$ are contravariant functorial in X. To see this we need to check that for a morphism $f : X \to Y$ we can construct a natural map $\mathbb{Z}(q)_Y \to f_*\mathbb{Z}(q)_X$. But this is true for any complex C of presheaves on Sm/k: for each open $U \subset Y$, the restriction $f^{-1}U \to U$ induces the desired map from $C_Y(U) = C(U)$ to $f_*C_X(U) = C(f^{-1}U)$.

The groups $H^{p,q}(X, A)$ are also covariantly functorial in k. That is, if $i : k \subset F$ is a field extension, there is a natural map $H^{*,*}(X, A) \to H^{*,*}(X_F, A)$. It is induced by the sheaf map $\mathbb{Z}(q)_X \to i_*\mathbb{Z}(q)_{X_F}$ assembled from the natural maps $\mathbb{Z}_{tr}(Y)(U) \to i_*\mathbb{Z}_{tr}(Y_F)(U) = \mathbb{Z}_{tr}(Y_F)(U_F)$ of exercise 1.12.

Proposition 3.7. *If $k \subset F$ is a finite and separable field extension and U is smooth over F, then the motivic complexes $\mathbb{Z}(q)_U$ are independent of the choice of the ground field (k or F). Hence the motivic cohomology groups $H^{p,q}(U, A)$ are independent of the choice of the ground field.*

Proof. Let T be any smooth scheme over k, and T_F its base change over F. By exercise 1.12 the groups $C_n\mathbb{Z}_{tr}(T_F)(U) = Cor_F(U \times_F \Delta^n_F, T_F)$ and $C_*\mathbb{Z}_{tr}(T)(U) = Cor_k(U \times_k \Delta^n_k, T)$ are isomorphic. That is, $C_*\mathbb{Z}_{tr}(T_F)(U) \cong C_*\mathbb{Z}_{tr}(T)(U)$. Letting T be $(\mathbb{A}^1_k - \{0\})^q$, the result follows from lemma 2.12, which says that the complex $\mathbb{Z}(q)[q]$ is a direct summand of $C_*(\mathbb{Z}_{tr}(T))$ over k, and of $C_*(\mathbb{Z}_{tr}(T_F))$ over F. $\square$

The following colimit lemmas are elementary consequences of exercise 1.13. They will be useful later on.

Lemma 3.8. *(Colimits) Let $k \subset F$ be a field extension and X smooth over k. Then:*

$$H^{*,*}(X_F, A) = \operatorname*{colim}_{\substack{k\subset E\subset F \\ E \text{ of finite type}}} H^{*,*}(X_E, A).$$

If $f : X \to S$ is a smooth morphism of smooth schemes over k such that S is connected and $F = k(S)$, then:

$$H^{*,*}(X \times_S \operatorname{Spec} F, A) = \operatorname*{colim}_{\substack{U\subset S \\ nonempty}} H^{*,*}(X \times_S U, A).$$

And now we want to introduce a multiplicative structure on the sheaves $\mathbb{Z}(n)$. We will need the following construction:

Construction 3.9. If (X_s, x_s) are pointed schemes for $s = 1, \ldots, j$, then for every $i < j$ we define a morphism of presheaves with transfers:

$$\mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_i) \otimes \mathbb{Z}_{tr}(X_{i+1} \wedge \ldots \wedge X_j) \to \mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_j).$$

Indeed, definition 1.9 provides a map:

$$\mathbb{Z}_{tr}(X_1 \times \ldots \times X_i)(U) \otimes \mathbb{Z}_{tr}(X_{i+1} \times \ldots \times X_j)(U)$$
$$= Cor_k(U, X_1 \times \ldots \times X_i) \otimes Cor_k(U, X_{i+1} \times \ldots \times X_j) \to$$
$$\to Cor_k(U \times U, X_1 \times \ldots \times X_j) = \mathbb{Z}_{tr}(X_1 \times \ldots \times X_j)(U \times U).$$

Composing with the diagonal $U \to U \times U$, we have:

$$\mathbb{Z}_{tr}(X_1 \times \ldots \times X_i)(U) \otimes \mathbb{Z}_{tr}(X_{i+1} \times \ldots \times X_j)(U) \xrightarrow{\Delta} \mathbb{Z}_{tr}(X_1 \times \ldots \times X_j)(U).$$

Now recall that by definition $\mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_n)$ is a quotient of $\mathbb{Z}_{tr}(X_1 \times \ldots \times X_n)$. It is easy to check that the map Δ factors through the quotient, giving the required morphism.

Construction 3.10. For each m and n we construct a map

$$\mathbb{Z}(m) \otimes \mathbb{Z}(n) \to \mathbb{Z}(m+n)$$

using the map $\mathbb{Z}_{tr}(G_m^{\wedge m}) \otimes \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m+n})$ of 3.9, as follows.

For any smooth U we need to build a map of complexes of abelian groups:

$$\mathbb{Z}(m)[m](U) \otimes \mathbb{Z}(n)[n](U) \to \mathbb{Z}(m+n)[m+n](U),$$

or equivalently, $\mathbb{Z}(m)(U) \otimes \mathbb{Z}(n)(U) \to \mathbb{Z}(m+n)(U)$. Recall that by definition 3.1, $\mathbb{Z}(n)[n](U)$ is the chain complex $C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(U)$. Let us write the underlying simplicial object as $A^n_\bullet = \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(U \times \Delta^\bullet)$, and the associated unnormalized chain complex $\mathbb{Z}(n)[n]$ as A^n_*. Similarly, we write $(A^m_\bullet \otimes A^n_\bullet)_*$ for the chain complex associated to $\mathrm{diag}(A^m_\bullet \otimes A^n_\bullet)$. The Eilenberg-Zilber theorem ([Wei94, 8.5.1]) yields a quasi-isomorphism $\nabla : A^m_* \otimes A^n_* \to (A^m_\bullet \otimes A^n_\bullet)_*$.

Therefore if we find a simplicial map $m : \mathrm{diag} A^m_\bullet \otimes A^n_\bullet \to A^{m+n}_\bullet$ we have also a map $(A^m_\bullet \otimes A^n_\bullet)_* \to A^{m+n}_*$ which, composed with the previous one, gives the multiplicative structure. Unfolding the definitions again, we have:

$$A^n_i = \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(U \times \Delta^i).$$

We define the components of m to be the maps of 3.9:

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m})(U \times \Delta^i) \otimes \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(U \times \Delta^i) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge (m+n)})(U \times \Delta^i).$$

The morphisms in 3.9 are associative and the map ∇ in the Eilenberg-Zilber theorem is homotopy associative ([Wei94, 8.5.4]). It follows that the pairing of construction 3.10 is homotopy associative.

Corollary 3.11. *For each smooth X, there are pairings:*

$$H^{p,q}(X,\mathbb{Z}) \otimes H^{p',q'}(X,\mathbb{Z}) \to H^{p+p',q+q'}(X,\mathbb{Z}).$$

In 15.9 we will show that this pairing is skew-commutative with respect to the first grading, so that $H^{*,*}(X,\mathbb{Z})$ is an associative graded-commutative ring.

Lecture 4

Weight one motivic cohomology

Theorem 4.1. *There is a quasi-isomorphism of complexes of presheaves with transfers:*

$$\mathbb{Z}(1) \xrightarrow{\simeq} \mathcal{O}^*[-1].$$

Corollary 4.2. *Let X be a smooth scheme over k. Then we have:*

$$H^{p,q}(X,\mathbb{Z}) = \begin{cases} 0 & q \leq 1 \text{ and } (p,q) \neq (0,0),(1,1),(2,1) \\ \mathbb{Z}(X) & (p,q) = (0,0) \\ \mathcal{O}^*(X) & (p,q) = (1,1) \\ \operatorname{Pic}(X) & (p,q) = (2,1) \end{cases}$$

This theorem will follow from lemmas $4.3 - 4.6$ below. An alternative proof is given in [SV96].

Consider the functor $\mathcal{M}^*(\mathbb{P}^1; 0, \infty) : Sm/k \to \mathbf{Ab}$ which sends a scheme X to the group of rational functions on $X \times \mathbb{P}^1$ which are regular in a neighborhood of $X \times \{0, \infty\}$ and equal 1 on $X \times \{0, \infty\}$. Clearly $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)$ is a sheaf for the Zariski topology. Given a rational function f on $X \times \mathbb{P}^1$ let $D(f)$ denote its divisor.

Lemma 4.3. *For all f in $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)(X)$, the divisor $D(f)$ belongs to the subgroup $Cor(X, \mathbb{A}^1 \setminus \{0\})$ of the group of cycles on $X \times \mathbb{P}^1$.*

From the lemma we get a morphism of sheaves:

$$\mathcal{M}^*(\mathbb{P}^1; 0, \infty) \hookrightarrow \mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\}).$$

$$
\begin{array}{cc|cccc}
 & & q & & & \\
H^{-2,2} & H^{-1,2} & H^{0,2} & H^{1,2} & H^{2,2} & H^{3,2} \\
0 & 0 & 0 & \mathcal{O}^*(X) & \mathrm{Pic}(X) & 0 \\
0 & 0 & \mathbb{Z}(X) & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \quad p
\end{array}
$$

Figure 4.1: Weight q motivic cohomology

Lemma 4.4. *For any connected X there is a short exact sequence in* **Ab***:*

$$0 \longrightarrow \mathcal{M}^*(\mathbb{P}^1; 0, \infty)(X) \longrightarrow \mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\})(X) \xrightarrow{\lambda} \mathbb{Z} \oplus \mathcal{O}^*(X) \longrightarrow 0.$$

Proof. We know that $\mathrm{Pic}(X \times \mathbb{P}^1) \cong \mathrm{Pic}(X) \times \mathbb{Z}$, so for any Z in $Cor(X, \mathbb{A}^1) \subset Cor(X, \mathbb{P}^1)$ there is a unique rational function f on $X \times \mathbb{P}^1$ and an integer n so that $D(f) = Z$ and $f/t^n = 1$ on $X \times \{\infty\}$. If Z lies in $Cor(X, \mathbb{A}^1 \setminus \{0\})$, then $f(0) \in \mathcal{O}^*(X)$. We define $\lambda : \mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\}) \to \mathbb{Z} \oplus \mathcal{O}^*$ by $\lambda(Z) = (n, (-1)^n f(0))$. If $u \in \mathcal{O}^*(X)$ and $Z_u = D(t - u)$ then $\lambda(Z_u) = (1, u)$. Since $\lambda(Z_u - Z_1) = (0, u)$ λ is onto. The kernel of λ consists of all Z whose f lies in $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)(X)$, so we are done. □

Lemma 4.5. *The map λ respects transfers. Hence $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)$ is a* **PST***.*

Proof. It is easy to see that the first component of λ is a morphism in **PST** because it is the map $Cor_k(X, \mathbb{A}^1 \setminus \{0\}) \to Cor_k(X, \mathrm{Spec}\, k)$, induced by the structure map $\pi : \mathbb{A}^1 \setminus \{0\} \to \mathrm{Spec}\, k$. To check the second component of λ, we see from exercise 1.13 that it suffices to check that the following diagram commutes for every finite field extension $F \subset E$.

$$
\begin{tikzcd}
\mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\})(\mathrm{Spec}\, E) \arrow[r] \arrow[d] & E^* \arrow[d, "N_{E/F}"] \\
\mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\})(\mathrm{Spec}\, F) \arrow[r] & F^*
\end{tikzcd}
$$

This is a straightforward verification using exercise 1.10. $\square$

Write $\mathcal{M}^*$ for $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)$. By 2.13, $(C_i F)(U) = F(U \times \Delta^i)$, so 4.4 gives us:

$$0 \to C_*(\mathcal{M}^*) \to C_*\mathbb{Z}_{tr}(\mathbb{A}^1 \setminus \{0\}) \to C_*(\mathbb{Z} \oplus \mathcal{O}^*) \to 0.$$

Splitting off $0 \to C_*\mathbb{Z} = C_*\mathbb{Z} \to 0$ we get an exact sequence:

$$0 \to C_*(\mathcal{M}^*) \to \mathbb{Z}(1)[1] \to C_*(\mathcal{O}^*) \to 0.$$

But $C_*(\mathcal{O}^*) \simeq \mathcal{O}^*$ because $\mathcal{O}^*(U \times \Delta^n) = \mathcal{O}^*(U)$. We will prove in lemma 4.6 that the first term $C_*(\mathcal{M}^*)$ is acyclic. Therefore $\mathbb{Z}(1)[1]$ is quasi-isomorphic to $\mathcal{O}^*$. This is the statement of the theorem 4.1, shifted once.

Lemma 4.6. *If X is a smooth scheme over k, then $C_*(\mathcal{M}^*)(X)$ is an acyclic complex of abelian groups. Hence $C_*(\mathcal{M}^*)$ is an acyclic complex of sheaves.*

Proof. Let $f \in C_i^{DK}(\mathcal{M}^*)(X)$ be a cycle, i.e., an element vanishing in $C_{i-1}^{DK}(\mathcal{M}^*)(X)$. Then f is a regular function on some neighborhood U of $Z = X \times \Delta^i \times \{0, \infty\}$ in $X \times \Delta^i \times \mathbb{P}^1$, and $f = 1$ on each face $X \times \Delta^{i-1} \times \mathbb{P}^1$, as well as on Z. Consider the regular function $h_X(f) = 1 - t(1 - f)$ on the neighborhood $\mathbb{A}^1 \times U$ of $\mathbb{A}^1 \times Z$ in $\mathbb{A}^1 \times X \times \Delta^i \times \mathbb{P}^1$, where t denotes the coordinate function of $\mathbb{A}^1$. Then $h_X(f)$ is a cycle in $C_i^{DK}(\mathcal{M}^*)(\mathbb{A}^1 \times X)$, because it equals 1 where f equals 1. The restrictions along $t = 0, 1$, from $C_i^{DK}(\mathcal{M}^*)(\mathbb{A}^1 \times X)$ to $C_i^{DK}(\mathcal{M}^*)(X)$, send $h_X(f)$ to 1 and f, respectively. Since these restrictions are chain homotopy equivalent by 2.17, f is a boundary. $\square$

This completes the proof of theorem 4.1.

Remark 4.7. We will revisit this in lecture 7 in 7.11.

Lemma 4.6 works more generally to show that $C_*\mathcal{M}^*(Y; Z)(X)$ is acyclic for every affine X, where $\mathcal{M}^*(Y; Z)(X)$ is the group of rational functions on $X \times Y$ which are regular in a neighborhood of $X \times Z$ and equal to 1 on $X \times Z$.

Now let us consider the complex $\mathbb{Z}/l(1)$. By theorem 4.1 $\mathbb{Z}(1)$ is quasi-isomorphic to $\mathcal{O}^*[-1]$. Tensoring with $\mathbb{Z}/l$ we have $\mathbb{Z}/l(1) \simeq \mathcal{O}^*[-1] \otimes^L \mathbb{Z}/l$, which is just the complex $[\mathcal{O}^* \xrightarrow{l} \mathcal{O}^*]$ in degrees 0 and 1. Then we have the universal coefficients sequence:

$$0 \longrightarrow H^{p,q}(X, \mathbb{Z})/l \longrightarrow H^{p,q}(X, \mathbb{Z}/l) \longrightarrow {}_lH^{p+1,q}(X, \mathbb{Z}) \longrightarrow 0.$$

Corollary 4.8. *There is a quasi-isomorphism of complexes of étale sheaves*

$$\mathbb{Z}/l(1)_{\acute{e}t} \simeq \mu_l.$$

Proof. Since sheafification is exact ([Mil80] p. 63), theorem 4.1 gives $\mathbb{Z}(1)_{\acute{e}t} \simeq \mathcal{O}^*_{\acute{e}t}[-1]$, and hence

$$\mathbb{Z}/l(1)_{\acute{e}t} \simeq \mathcal{O}^*_{\acute{e}t}[-1] \otimes^{\mathbb{L}} \mathbb{Z}/l \simeq \mu_l. \qquad \square$$

Corollary 4.9. *If $1/l \in k$ and X is smooth, then $H^{p,1}(X, \mathbb{Z}/l) = 0$ for $p \neq 0, 1, 2$ while:*

$$H^{0,1}(X, \mathbb{Z}/l) = \mu_l(X), \quad H^{1,1}(X, \mathbb{Z}/l) = H^1_{\acute{e}t}(X, \mu_l),$$

$$H^{2,1}(X, \mathbb{Z}/l) = \mathrm{Pic}(X)/l\,\mathrm{Pic}(X).$$

Proof. The calculation of $H^{p,1}$ for $p \neq 1$ follows from the universal coefficients sequence, since the only nonzero Zariski cohomology groups of $\mathcal{O}^*$ on a smooth scheme are H^0 and $H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$. For $p = 1$ note that corollary 4.8 gives a natural map

$$\mathbb{H}^*_{Zar}(X, \mathbb{Z}/l(1)) \to \mathbb{H}^*_{\acute{e}t}(X, \mathbb{Z}/l(1)_{\acute{e}t}) = H^1_{\acute{e}t}(X, \mu_l)$$

fitting into the diagram:

$$\begin{tikzcd}
H^1_{Zar}(X, \mathbb{Z}(1))/l \arrow[r, hook] \arrow[d, "\cong"'] & H^1_{Zar}(X, \mathbb{Z}/l(1)) \arrow[r, two heads] \arrow[d] & {}_lH^2_{Zar}(X, \mathbb{Z}(1)) \arrow[d, "\cong"] \\
H^1_{\acute{e}t}(X, \mathbb{Z}(1))/l \arrow[r, hook] & H^1_{\acute{e}t}(X, \mathbb{Z}/l(1)) \arrow[r, two heads] & {}_lH^2_{\acute{e}t}(X, \mathbb{Z}(1)).
\end{tikzcd}$$

Since $H^1_{\acute{e}t}(X, \mathcal{O}^*) = H^1_{Zar}(X, \mathcal{O}^*)$ by Hilbert's Theorem 90 (see [Mil80, III 4.9]), the 5-lemma concludes the proof. $\square$

Remark 4.10. (Deligne) If $\mathrm{char}\, k = l$ then $H^{1,1}(X, \mathbb{Z}/l) \cong H^1_{\mathrm{ffp}}(X, \mu_l)$. In fact, the proof of 4.9 is valid in this setting.

Lecture 5

Relation to Milnor K-Theory

The Milnor K-theory $K_*^M(F)$ of a field F is defined to be the quotient of the tensor algebra $T(F^*)$ over $\mathbb{Z}$ by the ideal generated by the elements of the form $x \otimes (1-x)$ where $x \in F^*$. In particular, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^*$.

The goal of this lecture is to prove the following:

Theorem 5.1. *For any field F and any n we have:*

$$H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \cong K_n^M(F).$$

We have already seen that this holds for $n = 0, 1$, because by definition 3.4 $H^{0,0}(\operatorname{Spec} F, \mathbb{Z}) = H^0_{Zar}(\operatorname{Spec} F, \mathbb{Z}) = \mathbb{Z}$ and by theorem 4.1:

$$H^{1,1}(\operatorname{Spec} F, \mathbb{Z}) = H^1_{Zar}(\operatorname{Spec} F, \mathcal{O}^*[-1]) = H^0_{Zar}(\operatorname{Spec} F, \mathcal{O}^*) = F^*.$$

The proof of theorem 5.1 will follow [SV00, 3.4] which is based on [NS89]. It will consist of three steps:

1. Construction of $\theta : H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \to K_n^M(F)$. This will use lemma 5.5.

2. Construction of $\lambda_F : K_n^M(F) \to H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$. This will be done using lemmas 5.9 and 5.6. The proof of lemma 5.9 will need lemma 5.8.
$$5.6 + (5.8 \Rightarrow 5.9) \Rightarrow \exists\ \lambda_F$$

3. Proof that these two maps are inverse to each other. For this we will need lemma 5.10 (proved using lemma 5.11).

Before starting the proof of the theorem we need some additional properties of motivic cohomology and Milnor K-theory.

Recall that $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F)$ is a quotient of $\mathbb{Z}_{tr}((\mathbb{A}^1 - \{0\})^n)(\operatorname{Spec} F)$, which by 1.10 is the group of zero cycles of $(\mathbb{A}^1 \setminus \{0\})^n$.

Lemma 5.2. *We have* $H^{p,q}(\operatorname{Spec} F, \mathbb{Z}) = H_{q-p}\big(C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))(\operatorname{Spec} F)\big)$ *for all* p *and* q. *In particular we have*

$$\begin{aligned} H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) =& H_0\left(C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(\operatorname{Spec} F)\right) \\ &= \operatorname{coker}\left(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F)\right). \end{aligned}$$

Proof. Write A_* for $C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})(\operatorname{Spec} F)$ so the right side is $H_{q-p}A_* = H^{p-q}A_*$. By definition 3.1, the restriction of $\mathbb{Z}(q)$ to $\operatorname{Spec} F$ is the chain complex $A_*[-q]$. Since Zariski cohomology on $\operatorname{Spec} F$ is just ordinary cohomology, we have

$$H^{p,q}(\operatorname{Spec} F, \mathbb{Z}) = H^p(A_*[-q]) = H^{p-q}(A_*) = H_{q-p}(A_*). \qquad \square$$

Lemma 5.3. *If* $F \subset E$ *is a finite field extension, then the proper push-forward of cycles induces a map* $N_{E/F} : H^{*,*}(\operatorname{Spec} E, \mathbb{Z}) \to H^{*,*}(\operatorname{Spec} F, \mathbb{Z})$. *Moreover, if* $x \in H^{*,*}(\operatorname{Spec} E, \mathbb{Z})$ *and* $y \in H^{*,*}(\operatorname{Spec} F, \mathbb{Z})$ *then:*

1. $N_{E/F} : H^{0,0}(\operatorname{Spec} E, \mathbb{Z}) = \mathbb{Z} \to \mathbb{Z} = H^{0,0}(\operatorname{Spec} F, \mathbb{Z})$ *is multiplication by the degree of* E/F.
2. $N_{E/F} : H^{1,1}(\operatorname{Spec} E, \mathbb{Z}) = E^* \to F^* = H^{1,1}(\operatorname{Spec} F, \mathbb{Z})$ *is the classical norm map* $E^* \to F^*$.
3. $N_{E/F}(y_E \cdot x) = y \cdot N_{E/F}(x)$ *and* $N_{E/F}(x \cdot y_E) = N_{E/F}(x) \cdot y$.
4. *If* $F \subset E \subset K$, *and* K *is normal over* F, *we have:*
$$N_{E/F}(x)_K = [E:F]_{insep} \sum_{j:E \hookrightarrow K} j^*(x) \qquad \text{in } H^{*,*}(\operatorname{Spec} K, \mathbb{Z}).$$
5. *If* $F \subset E' \subset E$ *then* $N_{E/F}(x) = N_{E'/F}(N_{E/E'}(x))$.

Proof. All but property 2 follow immediately from the corresponding properties of proper push-forward. Property 2 follows from property 4 since this formula also holds for the classical norm map $N_{E/F} : E^* \to F^*$. $\square$

If $F \subset E$ is a finite field extension, there is a "norm map" $N_{E/F} : K_n^M(E) \to K_n^M(F)$ satisfying the analogue of lemma 5.3. In addition, it satisfies the following condition (see [Sus82]).

Theorem 5.4 (Weil Reciprocity). *Suppose that L is an algebraic function field over k. For each discrete valuation w on L there is a map*

$$\partial_w : K_{n+1}^M(L) \to K_n^M(k(w)),$$

and for all $x \in K_{n+1}^M(L)$:

$$\sum_w N_{k(w)/k}\partial_w(x) = 0.$$

Corollary 5.5. *Let $p : Z \to \mathbb{A}_F^1$ be a finite surjective morphism and suppose that Z is integral. Let $f_1, \ldots, f_n \in \mathcal{O}^*(Z)$ and:*

$$p^{-1}(\{0\}) = \coprod n_i^0 z_i^0 \qquad p^{-1}(\{1\}) = \coprod n_i^1 z_i^1$$

where n_i^ϵ are the multiplicities of the points $z_i^\epsilon = \operatorname{Spec} E_i^\epsilon$ ($\epsilon = 0, 1$). Define:

$$\varphi_0 = \sum n_i^0 N_{E_i^0/F}(\{f_1, \ldots, f_n\}_{E_i^0}) \qquad \varphi_1 = \sum n_i^1 N_{E_i^1/F}(\{f_1, \ldots, f_n\}_{E_i^1})$$

then we have:

$$\varphi_0 = \varphi_1 \in K_n^M(F).$$

Proof. Let L be the function field of Z and consider $x = \{t/t-1, f_1, \ldots, f_n\}$. At every infinite place, $t/t-1$ equals 1 and $\partial_w(x) = 0$. Similarly, $\partial_w(x) = 0$ at all finite places except those over 0 and 1. If w_i lies over $t = 0$ then $\partial_{w_i}(x) = n_i^0\{f_1, \ldots, f_n\}$ in $K_n^M(E_i^0)$; if w_i lies over $t = 1$ then $\partial_{w_i}(x) = -n_i^1\{f_1, \ldots, f_n\}$ in $K_n^M(E_i^1)$. By Weil Reciprocity 5.4, $\sum N\partial_{w_i}(x) = \varphi_0 - \varphi_1$ vanishes in $K_n^M(F)$. □

We are now ready to define the map θ. By 5.2 it is enough to find a map f from $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F)$ to $K_n^M(F)$ which composed with the difference of the face operators is zero. Such a map must induce a unique map θ on the cokernel:

$$\begin{tikzcd} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \arrow[r, "\partial_0 - \partial_1"] & \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F) \arrow[r, two heads] \arrow[dr, "f"] & H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \arrow[d, "\theta"] \\ & & K_n^M(F). \end{tikzcd}$$

But now $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F)$ is a quotient of the free abelian group generated by the closed points of $(\mathbb{A}^1_F \setminus \{0\})^n$ (by exercise 1.10), modulo the subgroup generated by all points of the form $(x_1, \dots, 1, \dots, x_n)$ where the 1's can be in any position. If x is a closed point of $(\mathbb{A}^1_F \setminus \{0\})^n$ with residue field E then x is defined by a canonical sequence $(x_1, \dots, x_n)$ of nonzero elements of E. Now E is a finite field extension of F, and $\{x_1, \dots, x_n\} \in K_n^M(E)$. Using the norm map for Milnor K-theory $N_{E/F} : K_n^M(E) \to K_n^M(F)$, we define

$$f(x) = N_{E/F}(\{x_1, \dots, x_n\}).$$

Since $\{x_1, \dots, 1, \dots, x_n\} = 0$ in $K_*^M(E)$, this induces a well-defined map $f : \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F) \to K_n^M(F)$. By 5.5 the composition of f with the face operators is zero. We define θ to be the map induced on the cokernel.

If x is an F-point of $(\mathbb{A}^1_F - \{0\})^n$ then its coordinates $x_1, \dots, x_n$ are nonzero elements of F. We shall write $[x_1 : \cdots : x_n]$ for the class of x in $H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$. The map θ is obviously surjective since $\theta([x_1 : \cdots : x_n]) = \{x_1, \dots, x_n\}$ for $x_1, \dots, x_n$ in F.

Now let us build the opposite map, λ_F. For this, we will use the multiplicative structure (3.11) on $H^{*,*}(X, \mathbb{Z})$. The following lemma is immediate from the construction 3.10 and lemma 5.2.

Lemma 5.6. *For $a_1, \dots, a_n \in F$ we have $[a_1 : \cdots : a_n] = [a_1] \cdots [a_n]$.*

By definition $K_*^M(F) = T(F^*)/(x \otimes (1-x))$. Therefore we define a map:

$$T(F^*) \to \oplus_n H^{n,n}(\operatorname{Spec} F, \mathbb{Z}), \qquad a_1 \otimes \ldots \otimes a_n \mapsto [a_1] \cdots [a_n].$$

We will prove that this maps factors through $K_n^M(F)$. By 5.6, it is enough to prove that $[a : 1-a]$ is zero, which is the statement of the proposition 5.9 below.

Example 5.7. We can use a special cycle to show that $[a : -a] = 0$. Consider the correspondence Z from $\mathbb{A}^1$ (parametrized by t) to $X = \mathbb{A}^1 - \{0\}$ (parametrized by x) defined by

$$x^2 - t(a+b)x - (1-t)(1+ab)x + ab = 0.$$

Restricting along $t = 0, 1$ yields correspondences $[ab] + [1]$ and $[a] + [b]$ in $Cor(\operatorname{Spec} F, X)$. Setting these equal recovers the identity $[ab] = [a] + [b]$ in $H^{1,1}(\operatorname{Spec} F, \mathbb{Z}) \cong F^*$, because $[1] = 0$.

Let Y denote the composition of Z with the diagonal embedding $X \hookrightarrow X^2$. Since $[1:1] = [1][1] = 0$, equating the restrictions along $t = 0, 1$ yields the identity $[ab : ab] = [ab : ab] + [1 : 1] = [a : a] + [b : b]$ in $H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$. Bilinearity (5.6) yields skew-commutativity: $[a : b] + [b : a] = 0$. In particular, $2[a : a] = 0$.

Passing to $E = F(\sqrt{a})$, we see that $0 = 2[\sqrt{a} : \sqrt{a}] = [a : \sqrt{a}]$ in $H^{2,2}(\operatorname{Spec} E, \mathbb{Z})$. By 5.3, applying $N_{E/F}$ yields $0 = [a : -a]$ in $H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$.

Lemma 5.8. *Suppose $\exists n > 0$ so that $n[x : 1-x] = 0$ for all finite extensions of F and $x \neq 0, 1$ in F. Then $[x : 1-x] = 0$ in $H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$ for every $x \neq 0, 1$.*

Proof. Suppose $n = m{\cdot}p$ where p is a prime; we want to prove $m[x : 1-x] = 0$. Let us consider $y = \sqrt[p]{x}$ and $E = F(y)$. Then $0 = mp[y : 1-y] = m[x : 1-y]$, and $1 - x = N_{E/F}(1-y)$. Hence

$$0 = N_{E/F}(m[x : 1-x]) = m \cdot [x : N_{E/F}(1-y)] = m[x : 1-x].$$

The formula $[x : 1-x] = 0$ follows by induction on n. □

Proposition 5.9. *The element $[x : 1-x]$ in $H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$ is the zero element.*

Proof. Let Z be the finite correspondence from $\mathbb{A}^1$ (parametrized by t) to $X = \mathbb{A}^1 - \{0\}$ (parametrized by x) defined by:

$$x^3 - t(a^3+1)x^2 + t(a^3+1)x - a^3 = 0.$$

Let ω be a root of $x^2 + x + 1$, so $\omega^3 = 1$, and $E = F(\omega)$. The fiber over $t = 0$ consists of $a, \omega a$, and $\omega^2 a$ and the fiber over $t = 1$ consists of a^3 and two sixth roots of 1. Using the embedding $x \mapsto (x, 1-x)$ of $\mathbb{A}^1 - \{0, 1\}$ into X^2, Z yields a correspondence Z' from $\mathbb{A}^1$ to X^2. Then in $H^{2,2}(\operatorname{Spec} E, \mathbb{Z})$

$$\begin{aligned}\partial_0(Z') = [a : 1-a] + [\omega a : 1 - \omega a] + [\omega^2 a : 1 - \omega^2 a] = \\ [a : 1-a^3] + [\omega : (1-\omega a)(1-\omega^2 a)^2]\end{aligned}$$

is equal to

$$\partial_1(Z') = [a^3 : 1-a^3] + [-\omega : 1+\omega] + [-\omega^2 : 1+\omega^2].$$

Multiplying by 3 eliminates terms $[\omega : b]$, noting that $[-1 : 1+\omega] + [-1 : 1+\omega^2] = 0$ as $(1+\omega)(1+\omega^2) = 1$. Therefore $0 = 2[a^3 : 1-a^3]$ over E. Applying the norm yields $0 = 4[a^3 : 1-a^3]$ over F. Passing to the extension $F(\sqrt[3]{a})$ and norming yields $0 = 12[a : 1-a]$ over F. Applying lemma 5.8 with $n = 12$, we see that $0 = [a : 1-a]$ as well. □

Proposition 5.9 shows that the algebra map of lemma 5.6 induces a map on the quotient $\lambda_F : K_n^M(F) \to H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$. Now we need to check that λ_F and θ are inverse to each other. Since $\theta \circ \lambda_F$ is the identity by construction, it is enough to prove that λ_F is surjective.

Lemma 5.10. *The map λ_F is surjective.*

Proof. By 5.2, it suffices to show that if x is a closed point of $X = (\mathbb{A}^1_F \backslash \{0\})^n$ then $[x] \in H^{n.n}(\operatorname{Spec} F, \mathbb{Z})$ belongs to the image of λ_F. Set $E = k(x)$, and choose a lift $\tilde{x} \in X_E$ of x. Since x is the proper push-forward of $\tilde{x}$, the definition of the norm map (see 5.3) implies that:

$$[x] = N_{E/F}([\tilde{x}]) \qquad \tilde{x} = (a_1, \dots, a_n) \in (\mathbb{A}^1 \setminus \{0\})^n(E).$$

Since $\tilde{x}$ is a rational point of X_E, $[\tilde{x}]$ is the image under λ_E of its coordinates. So $[x] = N_{E/F}\lambda_E\{a_1, \dots, a_n\}$. The lemma now follows from the assertion, proven in 5.11 below, that the diagram (5.10.1) commutes. □

$$\begin{tikzcd} K_n^M(E) \arrow[r, "\lambda_E"] \arrow[d, "N_{E/F}"'] & H^{n,n}(\operatorname{Spec} E, \mathbb{Z}) \arrow[d, "N_{E/F}"] \\ K_n^M(F) \arrow[r, "\lambda_F"'] & H^{n,n}(\operatorname{Spec} F, \mathbb{Z}). \end{tikzcd} \tag{5.10.1}$$

Lemma 5.11. *If $F \subset E$ is any finite field extension, then the diagram (5.10.1) commutes.*

Proof. By 5.3 (3) we may assume that $[E : F] = l$ for some prime number l. Assume first that F has no extensions of degree prime to l and $[E : F] = l$. The Bass-Tate lemma (5.3) in [BT73] states that in this case $K_n^M(E)$ is generated by the symbols $a = \{a_1, \dots, a_{n-1}, b\}$ where $a_i \in F$ and $b \in E$. The properties of the norm on K_*^M and 5.6 yield:

$$\lambda_F N\{a_1, \dots, a_{n-1}, b\} = \lambda_F\{a_1, \dots, a_{n-1}, N(b)\} = [a_1 : \dots : a_{n-1}] \cdot [Nb].$$

But using the assertions of lemma 5.3 we have:

$$N\lambda_E(a) = N[a_1 : \dots : a_{n-1} : b] \overset{(2)}{=} [a_1 : \dots : a_{n-1}]\cdot N[b] \overset{(4)}{=} [a_1 : \dots : a_{n-1}]\cdot[Nb].$$

This concludes the proof in this case.

Now we use a standard reduction. For simplicity, we will write $H^{p,q}(F)$ for $H^{p,q}(\operatorname{Spec} F, \mathbb{Z})$. If F' is a maximal prime-to-l extension of F then the kernel of $H^{n,n}(F) \to H^{n,n}(F')$ is a torsion group of exponent prime to l by (1) and (3) of 5.3. Fix $a \in K_n^M(E)$. By the above case, $t = N\lambda_E(a) - \lambda_F N(a)$ is a torsion element of $H^{n,n}(F)$, of exponent prime to l.

Since the kernel of $H^{n,n}(F) \to H^{n,n}(E)$ has exponent l, $t_E \neq 0$ if and only if $t = 0$. If E is an inseparable extension of F then by 5.3(4) we have $t_E = l\lambda_E(a) - \lambda_E(la) = 0$. If E is separable over F then $E \otimes_F E$ is a finite product of fields E_i with $[E_i : E] < l$. Moreover, Weil Reciprocity implies that the diagrams

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{\text{diag}} & \oplus K_n^M(E_i) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle \oplus N_{E_i/E}} \\ K_n^M(F) & \longrightarrow & K_n^M(E) \end{array} \qquad \begin{array}{ccc} H^{n,n}(E) & \xrightarrow{\text{diag}} & \oplus H^{n,n}(E_i) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle \oplus N_{E_i/E}} \\ H^{n,n}(F) & \longrightarrow & H^{n,n}(E) \end{array}$$

commute (see p.387 of [BT73]). By induction on l, we have

$$t_E = \oplus N_{E_i/E}\lambda_{E_i}(a_{E_i}) - \oplus \lambda_E N_{E_i/E}(a_{E_i}) = 0.$$

Since $t_E = 0$ we also have $t = 0$. $\square$

This completes the proof of theorem 5.1.

Lecture 6

Étale sheaves with transfers

The goal of this lecture will be to study the relations between presheaves with transfers and étale sheaves. The main result (6.17) will be that sheafification preserves transfers.

Definition 6.1. A presheaf F of abelian groups on Sm/k is an **étale sheaf** if it restricts to an étale sheaf on each X in Sm/k. That is, if:

1. the sequence $0 \to F(X) \xrightarrow{\text{diag}} F(U) \xrightarrow{(+,-)} F(U \times_X U)$ is exact for every surjective étale morphism of smooth schemes $U \to X$;
2. $F(X \coprod Y) = F(X) \oplus F(Y)$ for all X and Y.

We will write $Sh_{\acute{e}t}(Sm/k)$ for the category of étale sheaves, which is a full subcategory of the category of presheaves of abelian groups.

A presheaf with transfers F is an **étale sheaf with transfers** if its underlying presheaf is an étale sheaf on Sm/k. We will write $Sh_{\acute{e}t}(Cor_k)$ for the full subcategory of $\mathbf{PST}(k)$ whose objects are the étale sheaves with transfers.

For example, we saw in lecture 2 that the étale sheaves $\mathbb{Z}$ and $\mathcal{O}^*$ have transfers, so they are étale sheaves with transfers. Lemma 6.2 shows that $\mathbb{Z}_{tr}(T)$ is an étale sheaf with transfers, even if T is singular (see 2.10).

Lemma 6.2. *For any scheme T over k, $\mathbb{Z}_{tr}(T)$ is an étale sheaf.*

Proof. Since $\mathbf{PST}(k)$ is an additive category, we have the required decomposition of $\mathbb{Z}_{tr}(T)(X \coprod Y) = \mathrm{Hom}_{\mathbf{PST}}(X \coprod Y, T)$. To check the sheaf axiom for surjective étale maps $U \to X$, we proceed as in the proof of 3.2.

As $U \times T \to X \times T$ is flat, the pullback of cycles is well-defined and is an injection. Hence the subgroup $\mathbb{Z}_{tr}(T)(X) = Cor_k(X,T)$ of cycles on $X \times T$ injects into the subgroup $\mathbb{Z}_{tr}(T)(U) = Cor_k(U,T)$ of cycles on $U \times T$.

To see that the sequence 6.1.1 is exact at $\mathbb{Z}_{tr}(T)(U)$, take Z_U in $Cor_k(U,T)$ whose images in $Cor_k(U \times_X U, T)$ coincide. We may assume that X and U are integral, and that the étale map $U \to X$ is finite; let F and L be their respective generic points. Then $Z_L \in Cor_F(L, T_F)$ comes from a cycle Z_F in $Cor_F(F, T_F)$ by 1.11, because if L lies in a Galois extension L' and $G = Gal(L'/F)$, then Z'_L lies in $Cor_F(L', T_F)^G = Cor_F(L, T_F)$. Thus by 1.13 there is a Zariski open $V \subset X$ and a cycle Z_V in $Cor_k(V,T)$ agreeing with Z_U in $Cor(U \times_X V, T)$. But U is finite and flat over X, so each term in Z_U is also finite over X. Hence each term in Z_V extends to a cycle in $X \times T$ finite over X, i.e., to a finite correspondence in $Cor_k(X,T)$. □

Corollary 6.3. *Let F be an étale sheaf with transfers. Then*

$$\mathrm{Hom}_{Sh_{\acute{e}t}(Cor_k)}(\mathbb{Z}_{tr}(X), F) = \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{tr}(X), F) = F(X).$$

Corollary 6.4. *For any abelian group A, the $A(n)$ are complexes of étale sheaves. If $1/n \in k$, the motivic complex of étale sheaves $\mathbb{Z}/n(1)$ is quasi-isomorphic to the étale sheaf μ_n.*

Proof. The $\mathbb{Z}(n)$ are étale sheaves with transfers by lemmas 2.12 and 6.2. We know that the $\mathbb{Z}_{tr}(T)$ are sheaves of free abelian groups. Hence $A \otimes \mathbb{Z}_{tr}(T)$ are étale sheaves. We conclude that the $A(n)$ are étale sheaves by the same argument we used for the $\mathbb{Z}(n)$. The last assertion is just a restatement of corollary 4.8 using 6.2. □

Exercise 6.5. Let $\pi : X \to S$ be a finite étale map, and π_t the induced finite correspondence from S to X. If F is any étale sheaf with transfers, show that $\pi_t^* : F(X) \to F(S)$ is the étale trace map of [Mil80, V.1.12]. *Hint:* If $Y \to S$ is Galois with group G, and factors through X, then $Cor(S,X) = Cor(Y,X)^G$ by 6.2. Show that the image of π in $Cor(Y,X)$ is the sum $\sum f$ of all S-maps from $f : Y \to X$, and hence determines $\pi_t \in Cor(S,X)$.

Locally constant étale sheaves form a second important class of étale sheaves with transfers.

Definition 6.6. The full subcategory Et/k of Sm/k consists of all the schemes of finite type over k which are smooth of dimension zero. Every

S in Et/k is a finite disjoint union of spectra of separable field extensions of k.

It is well known (see [Mil80] and [SGA4, VIII 2.2]) that the category of étale sheaves on Et/k is equivalent to the category of discrete modules over the profinite group $Gal(k_{sep}/k)$. If F corresponds to the Galois module M and $S = \mathrm{Spec}(\ell)$ then $F(S) = M^H$, where $H = Gal(k_{sep}/\ell)$.

We have the following functors:

$$Sh_{ét}(Et/k) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} Sh_{ét}(Sm/k),$$

where the restriction π_* is the right adjoint of π^*; they are both exact functors.

Definition 6.7. An étale sheaf is **locally constant** if $\pi^*\pi_*F \to F$ is an isomorphism. We will write $Sh^{lc}_{ét}$ for the full subcategory of $Sh_{ét}(Sm/k)$ consisting of all locally constant sheaves.

Exercise 6.8. Let F be the locally constant sheaf π^*M corresponding to the G-module M. If X is connected, and l is the separable closure of k in $H^0(X, \mathcal{O}_X)$, show that $F(X) = M^H$ where $H = Gal(k_{sep}/l)$. Conclude that π_*F is the Galois module M. Note that $F(X) = M^H$ is also defined if X is normal.

Lemma 6.9. *The functors π^* and π_* induce an equivalence between the category $Sh^{lc}_{ét}$ and the category of discrete modules over the profinite group $Gal(k_{sep}/k)$.*

Proof. If M is in $Sh_{ét}(Et/k)$, then $M \to \pi_*\pi^*M$ is an isomorphism by Ex. 6.8. Thus π^* is faithful. By category theory, $\pi^*\pi_*\pi^* \cong \pi^*$, so for F locally constant we have a natural isomorphism $\pi^*\pi_*F \cong F$. ☐

Exercise 6.10. Let L be a Galois extension of k, and let $G = Gal(L/k)$. Show that $\mathbb{Z}_{tr}(L)$ is the locally constant étale sheaf corresponding to the module $\mathbb{Z}[G]$.

Lemma 6.11. *Any locally constant étale sheaf has a unique underlying étale sheaf with transfers.*

Proof. Let $Z' \subset X \times Y$ be an elementary correspondence and let Z be the normalization of Z' in a normal field extension L of $F = k(X)$ containing $K = k(Z')$. If $G = Gal(L/F)$ then we also have $G = Aut_X(Z)$, and it is well known that the set $Hom_X(Z, Z')$ of maps $q : Z \to Z'$ over X is in one-one correspondence with the set of field maps $Hom_F(K, L)$. The cardinality of this set is the separable degree of K over F.

Let M be a Galois module, considered as a locally constant étale sheaf. It is easy to check using exercise 6.8 that $M(X)$ is isomorphic to $M(Z')^G$.

Write i for the inseparable degree of K over F. Then the transfer map $M(Y) \to M(X)$ is defined to be the composite of $M(Y) \to M(Z')$, multiplication by i, and the sum over all maps $q : Z \to Z'$ over X of $q^* : M(Z') \to M(Z)$.

The verification that this gives M the structure of a presheaf with transfers is now straightforward, and we refer the reader to 5.17 in [SV96] for details. □

It is clear that the locally constant étale sheaves form an abelian subcategory of $Sh_{ét}(Cor_k)$, i.e., the inclusion is an exact functor.

In order to describe the relation between presheaves and étale sheaves with transfers (see 6.18), we need two preliminary results.

If $p : U \to X$ is an étale cover, we define $\mathbb{Z}_{tr}(\check{U})$ to be the Čech complex

$$\dots \xrightarrow{p_0 - p_1 + p_2} \mathbb{Z}_{tr}(U \times_X U) \xrightarrow{p_0 - p_1} \mathbb{Z}_{tr}(U) \longrightarrow 0.$$

Proposition 6.12. *Let $p : U \to X$ be an étale covering of a scheme X. Then $\mathbb{Z}_{tr}(\check{U})$ is an étale resolution of the sheaf $\mathbb{Z}_{tr}(X)$, i.e., the following complex is exact as a complex of étale sheaves.*

$$\dots \xrightarrow{p_0 - p_1 + p_2} \mathbb{Z}_{tr}(U \times_X U) \xrightarrow{p_0 - p_1} \mathbb{Z}_{tr}(U) \xrightarrow{p} \mathbb{Z}_{tr}(X) \to 0$$

Proof. As this is a complex of sheaves it suffices to verify the exactness of the sequence at every étale point. Since points in the étale topology are strictly Hensel local schemes, it is enough to prove that, for every Hensel local scheme S over k, the following sequence of abelian groups is exact.

$$\cdots \to \mathbb{Z}_{tr}(U)(S) \to \mathbb{Z}_{tr}(X)(S) \to 0. \tag{6.12.1}$$

Here S is an inverse limit of smooth schemes S_i, and by abuse of notation $\mathbb{Z}_{tr}(T)(S)$ denotes $\lim \mathbb{Z}_{tr}(T)(S_i)$.

To prove that (6.12.1) is exact we need another reduction step. Let Z be a closed subscheme of $X \times S$ which is quasi-finite over S. We write $L(Z/S)$ for the free abelian group generated by the irreducible connected components of Z which are finite and surjective over S. $L(Z/S)$ is covariantly functorial on Z with respect to morphisms of quasi-finite schemes over S. Clearly, the sequence (6.12.1) is the colimit of complexes of the form:

$$\cdots \to L(Z_U \times_Z Z_U/S) \to L(Z_U/S) \to L(Z/S) \to 0 \qquad (6.12.2)$$

where $Z_U = Z \times_X U$ and the limit is taken over all Z closed subschemes of $X \times S$ which are finite and surjective over S. Therefore the proof of 6.12 will be completed once we show that the sequence (6.12.2) is exact for every subscheme Z of $X \times S$ which is finite and surjective over S.

Since S is Hensel local and Z is finite over S, Z is also Hensel. Therefore the covering $Z_U \to Z$ splits. Let $s_1 : Z \to Z_U$ be a splitting. We set $(Z_U)^k_Z = Z_U \times_Z \ldots \times_Z Z_U$. It is enough to check that the maps $s_k : L((Z_U)^k_Z/S) \to L((Z_U)^{k+1}_Z/S)$ are contracting homotopies where $s_k = L\left(s_1 \times_Z id_{(Z_U)^k_Z}\right)$.

This is the end of the proof of 6.12. $\square$

The proof shows that $\mathbb{Z}_{tr}(\check{U})$ is also a Nisnevich resolution of $\mathbb{Z}_{tr}(X)$, i.e., the sequence of 6.12 is also exact as a complex of Nisnevich sheaves. We can pinpoint why this proof holds in the étale topology and in the Nisnevich topology, but does not hold in the Zariski topology. This is because:

- If S is strictly Hensel local (i.e., a point in the étale topology) and Z is finite over S then Z is strictly Hensel.

- If S is Hensel local (i.e., a point in the Nisnevich topology) and Z is finite over S then Z is Hensel.

- If S is local (i.e., a point in the Zariski topology) and Z is finite over S then Z need *not* be local but will be semilocal.

Example 6.13. Let X be a connected semilocal scheme finite over a local scheme S. X is covered by its local subschemes U_i. If $X \to S$ does not split, its graph Γ defines an element of $\mathbb{Z}_{tr}(X)(S)$ that cannot come from $\oplus\mathbb{Z}_{tr}(U_i)(S)$, because Γ does not lie in any $S \times U_i$. Hence $\oplus\mathbb{Z}_{tr}(U_i) \to \mathbb{Z}_{tr}(X)$ is not a surjection of Zariski sheaves.

We will see in 13.13 that $\mathrm{Tot}(C_*\mathbb{Z}_{tr}(\check{U}))$ is a Zariski resolution of $C_*\mathbb{Z}_{tr}(X)$.

If $\mathcal{U} = \{U_i \to X\}$ is a Zariski covering, we can replace the infinite complex $\mathbb{Z}_{tr}(\check{U})$ of 6.12 by the bounded complex

$$\mathbb{Z}_{tr}(\check{\mathcal{U}}): \quad 0 \to \mathbb{Z}_{tr}(U_1 \cap \ldots \cap U_n) \to \ldots \to \oplus_i \mathbb{Z}_{tr}(U_i) \to 0.$$

Proposition 6.14. *Let $\mathcal{U} = \{U_i \to X\}$ be a Zariski open covering of X. Then $\mathbb{Z}_{tr}(\check{\mathcal{U}})$ is an étale resolution of $\mathbb{Z}_{tr}(X)$, i.e., the following sequence is exact as a complex of étale sheaves:*

$$0 \to \mathbb{Z}_{tr}(U_1 \cap \ldots \cap U_n) \to \ldots \to \oplus_i \mathbb{Z}_{tr}(U_i) \to \mathbb{Z}_{tr}(X) \to 0.$$

Proof. If $n = 2$, we apply 6.12 to $U = U_1 \coprod U_2$. Since $U \times_X U = U_1 \coprod (U_1 \cap U_2) \coprod U_2$, we see that the image of $\mathbb{Z}_{tr}(U^{\times 3})$ in $\mathbb{Z}_{tr}(U \times_X U)$ is $\mathbb{Z}_{tr}(U_1) \oplus \mathbb{Z}_{tr}(U_2)$ in the exact complex of 6.12. It follows that $\mathbb{Z}_{tr}(\check{\mathcal{U}}) \to \mathbb{Z}_{tr}(X)$ is exact for $n = 2$. For $n > 2$, the exactness follows by induction on n. $\square$

Example 6.15. If $\mathcal{U}$ is the cover of $\mathbb{P}^1$ by $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and $\operatorname{Spec} k[t^{-1}]$, and we mod out by the basepoint $t = 1$, we obtain the exact sequence

$$0 \to \mathbb{Z}_{tr}(\mathbb{G}_m) \to 2\mathbb{Z}_{tr}(\mathbb{A}^1, 1) \to \mathbb{Z}_{tr}(\mathbb{P}^1, 1) \to 0.$$

Applying C_* yields and exact sequence of complexes (see 2.13). Recalling that $C_*\mathbb{Z}_{tr}(\mathbb{A}^1, 1) \simeq 0$, we obtain quasi-isomorphisms of étale complexes (or even Nisnevich complexes)

$$C_*\mathbb{Z}_{tr}(\mathbb{P}^1, 1) \simeq C_*\mathbb{Z}_{tr}(\mathbb{G}_m)[1] = \mathbb{Z}(1).$$

Lemma 6.16. *Let $p : U \to Y$ be an étale covering and $f : X \to Y$ a finite correspondence. Then there is an étale covering $p' : V \to X$ and a finite correspondence $f' : V \to U$ so that the following diagram commutes in Cor_k.*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. We may suppose that f is defined by the elementary correspondence $Z \subset X \times Y$. Form the pullback $Z_U = Z \times_Y U$ inside $X \times U$. Since the projection $Z_U \to Z$ is étale and $Z \to X$ is finite, there is an étale cover $V \to X$ so that $V \times_X Z_U \to V \times_X Z$ has a section s. But then $s(V \times_X Z_U) \subset V \times U$ over V and defines the required finite correspondence $V \to U$. $\square$

$$
\begin{tikzcd}
V \times_X Z_U \arrow[r] \arrow[d] & Z_U \arrow[r, hook] \arrow[d] & X \times U \arrow[r] & U \arrow[d] \\
V \times_X Z \arrow[r] \arrow[d] \arrow[dr] & Z \arrow[r, hook] \arrow[d] & X \times Y \arrow[r] \arrow[dl] & Y \\
V \arrow[r] & X & &
\end{tikzcd}
$$

As in [Mil80] pp. 61-65, the inclusion $i : Sh_{ét}(Sm/k) \to PreSh(Sm/k)$ has a left adjoint $a_{ét}$, and $i \circ a_{ét}$ is left exact. Hence the category of étale sheaves on Sm/k is abelian, and the functor $a_{ét}$ is exact.

If F is a presheaf with transfers, the following theorem shows that its étale sheafification admits transfers. The same holds in the Nisnevich topology but *not* in the Zariski topology. However, we will prove later (in 21.15) that if F is a *homotopy invariant* presheaf with transfers, its Zariski sheafification admits transfers.

Recall that there is a forgetful functor $\varphi : \mathbf{PST}(k) \to PreSh(Sm/k)$.

Proposition 6.17. *Let F be a presheaf with transfers, and write $F_{ét}$ for $a_{ét}\varphi F$. Then $F_{ét}$ has a unique structure of presheaf with transfers such that $F \to F_{ét}$ is a morphism of presheaves with transfers.*

Corollary 6.18. *The inclusion functor $Sh_{ét}(Cor_k) \xhookrightarrow{i} \mathbf{PST}(k)$ has a left adjoint $a_{ét}$. The category $Sh_{ét}(Cor_k)$ is abelian, $a_{ét}$ is exact and commutes with the forgetful functor φ to (pre)sheaves on Sm/k.*

The connections between these abelian categories, given by 6.17 and 6.18, are described by the following diagram, where the φ are forgetful functors and both functors $a_{ét}$ are exact.

$$\begin{tikzcd} PreSh(Sm/k) \arrow[d, shift left, "a_{ét}"] & \mathbf{PST}(k) \arrow[l, "\varphi"'] \arrow[d, shift left, "a_{ét}"] \\ Sh_{ét}(Sm/k) \arrow[u, hook, shift left, "i"] & Sh_{ét}(Cor_k) \arrow[l, "\varphi"'] \arrow[u, hook, shift left, "i"] \end{tikzcd}$$

Proof of 6.17. Uniqueness. Suppose that two étale sheaves with transfers F_1 and F_2 satisfy the conditions of the theorem. We already know that $F_1(X) = F_2(X) = F_{ét}(X)$ for all X and we just need to check that $F_1(f) = F_2(f)$ holds when $f : X \to Y$ is a morphism in Cor_k. This is given if f comes from Sm/k.

Let $y \in F_1(Y) = F_2(Y) = F_{ét}(Y)$. Choose an étale covering $p : U \to Y$ so that $y|_U \in F_{ét}(U)$ is the image of some $u \in F(U)$. Lemma 6.16 yields the following diagram.

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Because $y|_U$ comes from $F(U)$, we have $F_1(f')(y|_U) = F_2(f')(y|_U)$.

$$\begin{aligned} F_1(p')F_1(f)(y) &= F_1(f')F_1(p)(y) && \text{as the diagram commutes,} \\ &= F_1(f')(y|_U) && \text{as } p \text{ comes from } Sm/k, \\ &= F_2(f')(y|_U) && \text{as } y|_U \text{ comes from } F(U), \\ &= F_2(p')F_2(f)(y) && \text{as the diagram commutes,} \\ &= F_1(p')F_2(f)(y) && \text{as } p' \text{ comes from } Sm/k. \end{aligned}$$

This implies that $F_1(f) = F_2(f)$ as p' is a covering and F_1 is an étale sheaf.

Existence. We need to define a morphism $F_{ét}(Y) \to F_{ét}(X)$ for each finite correspondence from X to Y. We first produce a map

$$F_{ét}(Y) \to Hom_{Sh}(\mathbb{Z}_{tr}(Y), F_{ét})$$

natural in Cor_k.

For all $y \in F_{ét}(Y)$ there is an étale covering $p : U \to Y$ and an element $u \in F(U)$ so that y and u agree in $F_{ét}(U)$. By representability (see 2.7), u determines a morphism $\mathbb{Z}_{tr}(U) \to F$ of presheaves with transfers. By shrinking U, we may arrange that the difference map sends u to zero in $F(U \times_X U)$. A chase in the commutative diagram below (where U_X^2 denotes $U \times_X U$) will produce the map of sheaves $[y] : \mathbb{Z}_{tr}(Y) \to F_{ét}$. The top row is exact by 6.12.

$$\begin{tikzcd}
0 \to Hom_{Sh}(\mathbb{Z}_{tr}(Y), F_{ét}) \arrow[r] & Hom_{Sh}(\mathbb{Z}_{tr}(U), F_{ét}) \arrow[r] & Hom_{Sh}(\mathbb{Z}_{tr}(U_X^2), F_{ét}) \\
& Hom_{\mathbf{PST}}(\mathbb{Z}_{tr}(U), F) \arrow[r] \arrow[u] & Hom_{\mathbf{PST}}(\mathbb{Z}_{tr}(U_X^2), F) \arrow[u]
\end{tikzcd}$$

We can now define a pairing $Cor(X,Y) \otimes F_{ét}(Y) \to F_{ét}(X)$. Let f be a correspondence from X to Y and $y \in F_{ét}(Y)$. By the map just described, y induces a morphisms of sheaves $[y] : \mathbb{Z}_{tr}(Y) \to F_{ét}$. Consider the composition:

$$\mathbb{Z}_{tr}(X) \xrightarrow{f} \mathbb{Z}_{tr}(Y) \xrightarrow{[y]} F_{ét}.$$

Hence there is a map $\mathbb{Z}_{tr}(X)(X) \to F_{ét}(X)$. The image of the identity map will be the pairing of f and y. $\square$

We conclude with an application of these ideas to homological algebra.

Proposition 6.19. *The abelian category $Sh_{ét}(Cor_k)$ has enough injectives.*

Proof. The category $\mathcal{S} = Sh_{ét}(Cor_k)$ has products and filtered direct limits are exact, because this is separately true for presheaves with transfers and for étale sheaves. That is, $\mathcal{S}$ satisfies axioms AB5 and AB3*. By 6.3, the family of sheaves $\mathbb{Z}_{tr}(X)$ is a family of generators of $\mathcal{S}$. It is well known (see [Gro57, 1.10.1]) that this implies that $\mathcal{S}$ has enough injectives. $\square$

Example 6.20. Let F be an étale sheaf with transfers. We claim that the terms $E^n(F)$ in its canonical flasque resolution (as an étale sheaf, see [Mil80] p. 90) are actually étale sheaves with transfers. For this it suffices to consider $E = E^0(F)$. Fix an algebraic closure $\bar{k}$ of k. For every X we define:

$$E(X) = \prod_{\bar{x} \in X(\bar{k})} F_{\bar{x}},$$

where $X(\bar{k})$ is the set of $\bar{k}$-points of X, and $F_{\bar{x}}$ denotes the fiber of F at $\bar{x}$. If $U \to X$ is étale, $E(U)$ is the product $\prod F_{\bar{x}}$ over $\bar{x} \in U(\bar{k})$. From this it follows that E is an étale sheaf, not only on X but on the big étale site of Sm/k. It is also easy to see that $F(X) \to E(X)$ is an injection.

In addition, E is a presheaf with transfers and $F \to E$ is a morphism in **PST**. For if $Z \subset X \times Y$ is an elementary correspondence from X to Y, we define the transfer $E(Y) \to E(X)$

$$E(Y) = \prod_{\bar{y} \in Y(\bar{k})} F_{\bar{y}} \to \prod_{\bar{x} \in X(\bar{k})} F_{\bar{x}} = E(X)$$

by stating that the component for $\bar{x} \in X(\bar{k})$ is the sum over all $\bar{y} \in Y(\bar{k})$ such that $z = (\bar{x}, \bar{y}) \in Z(\bar{k})$ of the localized transfers $F_{\bar{y}} \to F_{\bar{x}}$. To see that $F \to E$ is a morphism in **PST**, we may take X to be strictly Hensel local, so $F(X) = E(X)$. Since this forces Y to also be strictly Hensel semilocal, so $F(Y) = E(Y)$, this is a tautology.

The same construction works in the Nisnevich topology, letting $E(X)$ be the product over all closed points $x \in X$ of $F(\operatorname{Spec} \mathcal{O}^h_{X,x})$ (see 13.3). However, example 6.13 shows that it does not work in the Zariski topology, because the transfer $E(X) \to E(S)$ need not factor through the sum of the $E(U_i)$.

Lemma 6.21. *If F is any étale sheaf with transfers, then its cohomology presheaves $H^n_{ét}(-, F)$ are presheaves with transfers.*

Proof. The canonical flasque resolution $F \to E^*(F)$ of 6.20 is a resolution of sheaves with transfers. Since the forgetful functor from $\mathbf{PST}(k)$ to presheaves is exact, and $H^n(-, F)$ is the cohomology $E^*(F)$ as a presheaf, we see that $H^n(-, F)$ is also the cohomology of $E^*(F)$ in the abelian category $\mathbf{PST}(k)$. □

Example 6.22. By 2.4, $F = \mathbb{G}_m$ is an étale sheaf with transfers. By 6.21, both the Picard group $\operatorname{Pic}(X) = H^1_{ét}(X, \mathbb{G}_m)$ and the cohomological Brauer group $Br'(X) = H^2_{ét}(X, \mathbb{G}_m)_{tors}$ are presheaves with transfers.

Lemma 6.23. *For any $F \in Sh_{ét}(Cor_k)$ and any smooth X and $i \in \mathbb{Z}$ we have:*

$$Ext^i_{Sh_{ét}(Cor_k)}(\mathbb{Z}_{tr}(X), F) = H^i_{ét}(X, F).$$

Proof. The case $i = 0$ is $\operatorname{Hom}(\mathbb{Z}_{tr}(X), F) = F(X)$; this is 6.3. For $i > 0$ it suffices to show that if F is an injective étale sheaf with transfers then $H^i(X, F)$ is zero. Consider the canonical flasque resolution $E^*(F)$ of example 6.20. Since F is injective, the canonical inclusion $F \to E^0$ must split, i.e., F is a direct factor of E^0 in $Sh_{ét}(Cor_k)$. Since $H^i_{ét}(X, F)$ is a direct summand of $H^i(X, E^0)$, it must vanish for $i > 0$. □

If we restrict to the category $Sh_{ét}(Cor_k, R)$ of étale sheaves of R-modules with transfer, $E^0(F)$ is a flasque sheaf of R-modules with transfer by 6.20. The proof of 6.23 goes through word for word to prove the following variation.

Porism 6.24. *For any $F \in Sh_{ét}(Cor_k, R)$ and any smooth X and $i \in \mathbb{Z}$:*

$$Ext^i_{Sh_{ét}(Cor_k,R)}(R_{tr}(X), F) = H^i_{ét}(X, F).$$

The same proof also shows that lemmas 6.23 and 6.24 hold for the Nisnevich topology (see 13.4). See [TriCa, 3.1.8] for an alternative proof.

Exercise 6.25. Still assuming that $\mathrm{cd}_R(k) < \infty$, let K be any complex of étale sheaves of R-modules with transfer. Show that its hyperext and hypercohomology agree in the sense that for any smooth X and $i \in \mathbb{Z}$:

$$\mathrm{Ext}^i(R_{tr}(X), K) \cong \mathbb{H}^n_{ét}(X, K).$$

If one is willing to extend the constructions of this paper to possibly singular schemes, the *cdh* topology would play a major role. The constructions of this section can be carried out in the *cdh* topology as well.

Lecture 7

Relative Picard group and Suslin's Rigidity Theorem

In this lecture we introduce the relative Picard group $\mathrm{Pic}(\bar{X}, X_\infty)$. When $\bar{X}$ is a good compactification of X over S, its elements determine maps $F(X) \to F(S)$ for every homotopy invariant F. This pairing will be used to prove Suslin's Rigidity Theorem 7.20.

Recall from 1A.7 and 1A.8 that if S is a smooth connected scheme and $p : X \to S$ a smooth morphism then we write $c(X/S, 0)$ for the free abelian group generated by the irreducible closed subsets of X which are finite and surjective over S. In this lecture we will write $C_0(X/S)$ for $c(X/S, 0)$.

By 1A.10, given a map $S' \to S$, there is a map $C_0(X/S) \to C_0(X \times_S S'/S')$, induced from

$$C_0(X/S) \hookrightarrow \mathbb{Z}_{tr}(X)(S) = Cor_k(S, X).$$

Definition 7.1. We define $H_0^{sing}(X/S)$ to be the cokernel of the map

$$C_0(X \times \mathbb{A}^1/S \times \mathbb{A}^1) \xrightarrow{\partial_0 - \partial_1} C_0(X/S)$$

where ∂_i is induced by "$t = i$" : $\mathrm{Spec}\, k \to \mathbb{A}^1_k$.

Example 7.2. If $X = Y \times_k S$ then $C_0(X/S) = Cor(S, Y) = \mathbb{Z}_{tr}(Y)(S)$. In addition, $X \times \mathbb{A}^1 = Y \times_k (S \times \mathbb{A}^1)$ and the following diagram commutes:

$$\begin{tikzcd}
C_0(X \times \mathbb{A}^1/S \times \mathbb{A}^1) \arrow[r] \arrow[d, "="] & C_0(X/S) \arrow[d, "="] \\
\mathbb{Z}_{tr}(Y)(S \times \mathbb{A}^1) \arrow[r] & \mathbb{Z}_{tr}(Y)(S).
\end{tikzcd}$$

Taking cokernels, we conclude (using 2.26) that:

$$H_0^{sing}(Y \times S/S) = H_0 C_* \mathbb{Z}_{tr}(Y)(S) = Cor(S,Y)/\mathbb{A}^1\text{-homotopy}.$$

In particular, this implies that two elements of $Cor(S,Y)$ are $\mathbb{A}^1$-homotopic exactly when they agree in $H_0^{sing}(Y \times S/S)$.

If $S = \operatorname{Spec} k$ then $H_0^{sing}(X/S)$ is the cokernel $H_0^{sing}(X/k)$ of $\mathbb{Z}_{tr}(X)(\mathbb{A}^1) \to \mathbb{Z}_{tr}(X)(S)$ discussed in exercise 2.20, because $C_0(X/S) = \mathbb{Z}_{tr}(X)(\operatorname{Spec} k)$. Also by 2.20, there is a natural surjection $H_0^{sing}(X/S) \to CH_0(X)$. If X is projective, this surjection is an isomorphism.

Example 7.3. If $S = \operatorname{Spec} k$, then 7.16 below shows that $H_0^{sing}(\mathbb{P}^1/S) = H_0^{sing}(\mathbb{A}^1/S) = \mathbb{Z}$ but $H_0^{sing}(\mathbb{A}^1 - \{0\}/S) = \mathbb{Z} \oplus k^*$.

Remark 7.4. In [SV96] the groups $H_*^{sing}(X/S)$ are defined to be the homology of the evident chain complex $C_*(X/S)$ with

$$C_n(X/S) = C_0(X \times \Delta^n / S \times \Delta^n).$$

We will consider the singular homology $H_*^{sing}(X/S)$ in lecture 10 below when $S = \operatorname{Spec} k$, and $C_*(X/S) = C_* \mathbb{Z}_{tr}(X)(S)$.

Let F be a **PST**. The map $\mathrm{Tr} : C_0(X/S) \otimes F(X) \to F(S)$ is defined to be the inclusion $C_0(X/S) \subset Cor_k(S,X)$ (see 1A.10) followed by evaluation on $F(X)$.

$$\begin{tikzcd}
C_0(X/S) \otimes F(X) \arrow[r, "\mathrm{Tr}"] \arrow[d, hook] & F(S) \\
Cor_k(S,X) \otimes F(X) \arrow[ur, "\text{evaluate}"'] &
\end{tikzcd}$$

Lemma 7.5. *If F is homotopy invariant presheaf with transfers then the map Tr factors through $H_0^{sing}(X/S) \otimes F(X) \to F(S)$.*

Proof. Since $F(X) = F(X \times \mathbb{A}^1)$, we have a diagram

$$\begin{tikzcd}
C_0(X \times \mathbb{A}^1/S \times \mathbb{A}^1) \otimes F(X) \arrow[r, "\mathrm{Tr}"] \arrow[d, "\partial_0 - \partial_1"] & F(S \times \mathbb{A}^1) \arrow[d, "i_0 - i_1 = 0"] \\
C_0(X/S) \otimes F(X) \arrow[r, "\mathrm{Tr}"] & F(S).
\end{tikzcd}$$

□

Example 7.6. If $\sigma : S \to X$ is a section of p, regarded as an element of $H_0^{sing}(X/S)$, then $Tr(\sigma, -)$ is the usual map $\sigma^* : F(X) \to F(S)$.

Remark 7.7. The pairing $H_0^{sing}(X/S) \otimes F(X) \to F(S)$ is fundamental. It can be defined more generally for homotopy invariant presheaves equipped only with transfer maps $\mathrm{Tr}_D : F(X) \to F(S)$ for any relative smooth curve X/S and any effective divisor $D \subset X$ which is finite and surjective over S, such that the transfer maps form a "pseudo pretheory". This construction applies to the K-theory presheaves $K_n(X)$, equipped with the transfer maps of exercise 2.6, even though these are not presheaves with transfers.

In order to compute $H_0^{sing}(X/S)$, it is useful to embed X in a slightly larger scheme $\bar{X}$.

Definition 7.8. A smooth curve $p : X \to S$ admits a **good compactification** $\bar{X}$ if it factors as:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "p"'] & \bar{X} \arrow[d, "\bar{p}"] \\ & S \end{tikzcd}$$

where j is an open embedding, $\bar{X}$ is a proper normal but not necessarily smooth curve over S and $Y = \bar{X} - X$ has an affine open neighborhood in $\bar{X}$.

If S is affine, for example, then $X = \mathbb{A}^1 \times S$ admits $\mathbb{P}^1 \times S$ as a good compactification. Similarly, if C is any smooth affine curve over k then $C \times S \to S$ admits $\bar{C} \times S$ as a good compactification. The following result implies that every point x of every X has an open neighborhood U which has a good compactification over a generic projection $X \to \mathbb{A}^{l-1}$.

Lemma 7.9. *Let $p : X \to \mathbb{A}^l$ be an étale map. If k is infinite, there exists a linear projection $\mathbb{A}^l \to \mathbb{A}^{l-1}$ so that the composition $X \to \mathbb{A}^{l-1}$ is a curve with a good compactification.*

Proof. There is an open $U \subset \mathbb{A}^l$ so that X is quasi-finite and surjective over U. Choose a linear projection $\mathbb{A}^l \to \mathbb{A}^{l-1}$ so that the restriction to

$\mathbb{A}^l - U$ is finite; $\mathbb{A}^l$ has good compactification $Y = \mathbb{P}^1 \times \mathbb{A}^{l-1}$. By Zariski's Main Theorem (as formulated in [EGA4, 8.12.6]), the map $X \to Y$ may be factored as an open immersion $X \hookrightarrow \bar{X}$ followed by a finite map $\bar{p} : \bar{X} \to Y$. Replacing $\bar{X}$ by its normalization, we may assume that $\bar{X}$ is normal. Note that $\bar{p}$ is an affine map. Since Y is a good compactification of U, $\bar{X}$ is a good compactification of X. $\square$

Definition 7.10. If $Y \overset{i}{\hookrightarrow} \bar{X}$ is closed we set $\mathbb{G}_{\bar{X},Y} = Ker(\mathcal{O}^*_{\bar{X}} \to i_*\mathcal{O}^*_Y)$. The **relative Picard group** is defined to be:

$$\mathrm{Pic}(\bar{X}, Y) = H^1_{Zar}(\bar{X}, \mathbb{G}_{\bar{X},Y}).$$

By [Mil80] p. 124, we also have $\mathrm{Pic}(\bar{X}, Y) = H^1_{ét}(\bar{X}, \mathbb{G}_{\bar{X},Y})$.

By [SV96, 2.1], the elements of $\mathrm{Pic}(\bar{X}, Y)$ are the isomorphism classes $(\mathcal{L}, t)$ of line bundles $\mathcal{L}$ on $\bar{X}$ with a trivialization t on Y. The group operation is $\otimes$, i.e., $(\mathcal{L}, t) \otimes (\mathcal{L}', t') = (\mathcal{L} \otimes \mathcal{L}', t \otimes t')$.

Remark 7.11. For $\bar{X} = S \times \mathbb{P}^1$ and $Y = S \times \{0, \infty\}$, the "stalk" $(i^*\mathbb{G}_{\bar{X},Y})(Y)$ of $\mathbb{G}_{\bar{X},Y}$ at Y is the group $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)(S)$ of lecture 4.

The cohomology of $\mathcal{O}^* \to i_*\mathcal{O}^*_Y$ yields the exact sequence

$$\mathcal{O}^*(\bar{X}) \to \mathcal{O}^*(Y) \to \mathrm{Pic}(\bar{X}, Y) \to \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(Y).$$

Comparing this exact sequence for $\bar{X}$ and $\bar{X} \times \mathbb{A}^1$ yields:

Corollary 7.12. *If $\bar{X}$ is a normal scheme and Y is reduced, we have:*

$$\mathrm{Pic}(\bar{X}, Y) = \mathrm{Pic}(\bar{X} \times \mathbb{A}^1, Y \times \mathbb{A}^1).$$

Let us write j for the open embedding of $X = \bar{X} - Y$ into $\bar{X}$.

Lemma 7.13. *If $1/n \in k$, there is a natural injection*

$$\mathrm{Pic}(\bar{X}, Y)/n \hookrightarrow H^2_{ét}(\bar{X}, j_!\mu_n).$$

Proof. By Kummer Theory we have an exact sequence of étale sheaves:

$$0 \longrightarrow j_!\mu_n \longrightarrow \mathbb{G}_{\bar{X},Y} \xrightarrow{n} \mathbb{G}_{\bar{X},Y} \longrightarrow 0. \tag{7.13.1}$$

Applying étale cohomology yields:

$$H^1(\bar{X}, j_!\mu_n) \longrightarrow H^1(\bar{X}, \mathbb{G}_{\bar{X},Y}) \xrightarrow{n} H^1(\bar{X}, \mathbb{G}_{\bar{X},Y}) \longrightarrow H^2_{ét}(\bar{X}, j_!\mu_n).$$

But the middle groups are both $\mathrm{Pic}(\bar{X}, Y)$. $\square$

Example 7.14. Suppose that $S = \operatorname{Spec} k$ and k is algebraically closed. If $\bar{X}$ is a smooth connected curve, then $\operatorname{Pic}(\bar{X}, Y)$ is an extension of $\operatorname{Pic}(\bar{X})$ by a finite product of $|Y| - 1$ copies of k^*. Hence $\operatorname{Pic}(\bar{X}, Y)/n \cong H^2_{\acute{e}t}(\bar{X}, \mu_n) \cong \mathbb{Z}/n$.

Recall that $C_0(X/S)$ is generated by closed subsets Z of X which are finite and surjective over S. Because X is smooth, each such subset is an effective Cartier divisor on $\bar{X}$, and has an associated line bundle $\mathcal{L}$ equipped with a canonical map $\mathcal{O} \to \mathcal{L}$. This map gives a trivialization of $\mathcal{L}$ on $\bar{X} - Z$, which is a neighborhood of Y. Thus a good compactification $\bar{X}$ induces a homomorphism

$$C_0(X/S) \to \operatorname{Pic}(\bar{X}, Y).$$

When Y lies in an affine open neighborhood, this map is onto because every trivialization on Y extends to a neighborhood of Y.

Exercise 7.15. In this exercise we make the lifting to $C_0(X/S)$ explicit. Suppose that $\mathcal{L}$ is a line bundle on $\bar{X}$ with a fixed trivialization t on an open neighborhood U of Y. Show that t gives a canonical isomorphism of $\mathcal{L}$ with a Cartier divisor $\mathcal{L}(D)$, i.e., an invertible subsheaf of the sheaf $\mathcal{K}$ of total quotient rings of $\mathcal{O}$. (See [Har77, II.6].) Show that $\mathcal{L}(D)$ comes from a Weil divisor $D = \sum n_i Z_i$ on $\bar{X}$ with the Z_i supported on $\bar{X} - U$. Then show that the map $C_0(X/S) \to \operatorname{Pic}(\bar{X}, Y)$ sends $\sum n_i Z_i$ to $(\mathcal{L}, t)$.

Because $C_1(X/S) \to \operatorname{Pic}(\bar{X}, Y)$ factors through $\operatorname{Pic}(\bar{X} \times \mathbb{A}^1, Y \times \mathbb{A}^1)$, 7.12 shows that $C_0(X/S) \to Pic(\bar{X}, Y)$ induces a homomorphism

$$H_0^{sing}(X/S) \to \operatorname{Pic}(\bar{X}, Y).$$

Theorem 7.16. *Let S be a smooth scheme. If $p : X \to S$ is a smooth quasi-affine curve with a good compactification $(\bar{X}, Y)$, then:*

$$H_0^{sing}(X/S) \xrightarrow{\cong} \operatorname{Pic}(\bar{X}, Y).$$

Proof. The kernel of $C_0(X/S) \to \operatorname{Pic}(\bar{X}, Y)$ consists of $f \in K(\bar{X})$ which are defined and equal to 1 on Y. Since X is quasi-affine over S, Y contains at least one point in every irreducible component of every fiber of $\bar{X}$ over S. Therefore the divisor D of $tf + (1-t)$ defines an element of $C_0(X \times \mathbb{A}^1/S \times \mathbb{A}^1)$ with $\partial_0 D = 0$ and $\partial_1 D = (f)$. Hence (f) represents 0 in $H_0^{sing}(X/S)$. This proves that the map $H_0^{sing}(X/S) \to \operatorname{Pic}(\bar{X}, Y)$ is an injection, hence an isomorphism. $\square$

Theorem 7.16 also holds when X is not quasi-affine over S, but the proof is more involved.

Corollary 7.17. *If F is a homotopy invariant presheaf with transfers, there is a pairing*

$$\mathrm{Pic}(\bar{X}, Y) \otimes F(X) \to F(S).$$

Example 7.18. If X is a smooth curve over k and $1/n \in k$, then any two geometric points $x, x' : \operatorname{Spec} \bar{k} \to X$ induce the same map $F(X) \to F(\operatorname{Spec} \bar{k})$. Here F is any homotopy invariant presheaf with transfers satisfying $nF = 0$. Indeed, $[x] = [x']$ in $\mathrm{Pic}(\bar{X}, Y)/n$ by example 7.14. This phenomenon is known as "rigidity," and is a simple case of Theorem 7.20 below.

Corollary 7.19. *Let $p : X \to S$ be a smooth curve with a good compactification. Assume that S is Hensel local and let $X_0 \to S_0$ be the closed fiber of p. Then for every n prime to $\operatorname{char} k$ the following map is injective:*

$$H_0^{sing}(X/S)/n \to H_0^{sing}(X_0/S_0)/n.$$

Proof. Kummer Theory yields the exact sequence 7.13.1 of étale sheaves, and similarly for $(\bar{X}_0, Y_0)$. Applying étale cohomology yields:

$$\begin{tikzcd}
H^1(\bar{X}, j_!\mu_n) \arrow[r] \arrow[d] & H^1(\bar{X}, \mathbb{G}_{\bar{X},Y}) \arrow[r, "n"] \arrow[d] & H^1(\bar{X}, \mathbb{G}_{\bar{X},Y}) \arrow[r] \arrow[d] & H^2_{\acute{e}t}(\bar{X}, j_!\mu_n) \arrow[d, "\cong"] \\
H^1(\bar{X}_0, j_!\mu_n) \arrow[r] & H^1(\bar{X}_0, \mathbb{G}_{\bar{X},Y}) \arrow[r, "n"] & H^1(\bar{X}_0, \mathbb{G}_{\bar{X},Y}) \arrow[r] & H^2_{\acute{e}t}(\bar{X}_0, j_!\mu_n).
\end{tikzcd}$$

Since $H^2(\bar{X}, j_!\mu_n) = H^2_c(X, \mu_n)$, the right vertical map is an isomorphism by proper base change with compact supports (see [Mil80, VI.3.2]). We have a diagram:

$$\begin{tikzcd}
\mathrm{Pic}(\bar{X}, Y)/n \arrow[r, hook] \arrow[d] & H^2_{\acute{e}t}(\bar{X}, j_!\mu_n) \arrow[d, "\cong"] \\
\mathrm{Pic}(X_0, Y_0)/n \arrow[r, hook] & H^2_{\acute{e}t}(\bar{X}_0, j_!\mu_n).
\end{tikzcd}$$

Corollary 7.19 now follows from theorem 7.16. $\square$

It follows from 6.8 that every locally constant étale sheaf F is homotopy invariant, because $H^0(X \times \mathbb{A}^1, \mathcal{O}) \cong H^0(X, \mathcal{O}) \otimes_k k[t]$. The following result shows that the converse is true for torsion sheaves. (Cf. [SV96, 4.5].)

Theorem 7.20. *(Suslin's "Rigidity Theorem") Let F be a homotopy invariant presheaf with transfers, such that the groups $F(X)$ are torsion of exponent prime to* char k. *Then $F_{\acute{e}t}$ is locally constant.*

Proof. Let $F_0 = \pi_* \pi^*(F)$ be the locally constant sheaf for the group $M = F(k_{sep})$. We want to show that the adjunction $F_0 \to F$ is an isomorphism of étale sheaves. It suffices to check on stalks. Since $\mathcal{O}^{sh}_{X,x}$ contains a separable closure of k, we may assume that k is separably closed. In this case 7.20 asserts that $F_{\acute{e}t}$ is the constant sheaf for the group $M = F(\operatorname{Spec} k)$. Since X is smooth at x, $\mathcal{O}^{sh}_{X,x}$ is isomorphic to the Henselization of $\mathbb{A}^l$ at $\{0\}$. Thus the Rigidity Theorem is a consequence of proposition 7.21 below. ☐

Proposition 7.21. *Let S_l be the Henselization at $\{0\}$ in $\mathbb{A}^l$ over a separably closed field k. Assume that F is as is 7.20. Then $F(S_l) = F(\operatorname{Spec} k)$.*

Proof. The hypothesis on F is inherited by $F(X)_n = \{x \in F(X) : nx = 0\}$. Therefore we may assume that F has exponent n for some prime n.

We use the following sequence of inclusions:

$$\operatorname{Spec} k = S_0 \subset \ldots \subset S_{l-1} \overset{i}{\subset} S_l.$$

By induction on l, it is enough to prove that the map $F(i) : F(S_l) \to F(S_{l-1})$ is an isomorphism. For this it suffices to prove that $F(i)$ is an injection, because it is split by the projection π

$$S_{l-1} \underset{i}{\overset{\pi}{\leftrightarrows}} S_l.$$

But $F(S_l) = \operatorname{colim}_{(X,x_0) \to (\mathbb{A}^l,0)} F(X)$ where the colimit is taken over all diagrams:

$$S_l \xrightarrow{\pi} X \xrightarrow{p} \mathbb{A}_l.$$

It suffices to show for every X that if $\varphi \in F(X)$ has $i_l^* \pi^* \varphi = 0$ then $\pi^* \varphi = 0$. By lemma 7.9 there is a curve $X \to \mathbb{A}_{l-1}$ with a good compactification. Let

X' be the pullback in the following diagram:

$$\begin{tikzcd}
S_{l-1} \arrow[r, "i_l"] & S_l \arrow[dr, "s_1"] \arrow[drrr, "\pi"] \arrow[ddr, "Id"'] & & & \\
& & X' \arrow[rr, "q"] \arrow[d] & & X \arrow[d] \\
& & S_l \arrow[r, "\pi_l"] & S_{l-1} \arrow[r] & \mathbb{A}^{l-1}
\end{tikzcd}$$

The maps π and $\pi i_l \pi_l : S_l \to X$ induce two sections $s_1, s_2 : S_l \to X'$ of $X' \to S_l$ which agree on the closed fiber $X_0 = X \times_S S_0$. Given $\varphi \in F(X)$ we need to show that $\pi_l^* i_l^* \pi^* \varphi = \pi^* \varphi$. But $\pi^* \varphi = s_1^* q^*(\varphi)$ and $\pi_l^* i_l^* \pi^* \varphi = s_2^* q^*(\varphi)$. The s_i coincide on the closed point of S_l by construction. So we are left to prove that $s^*(\psi) = (s')^*(\psi)$ for all $\psi \in F(X')$ and any $s, s' : S_l \to X'$ with $s_0 = s'_0$. Consider the following diagram:

$$\begin{tikzcd}
(\Gamma_s - \Gamma_{s'}) \otimes \psi \arrow[rr, mapsto] & & s^*(\psi) - s'^*(\psi) \\
C_0(X'/S_l) \otimes F(X') \arrow[r, two heads] & H_0(X'/S_l) \otimes F(X') \arrow[r, "\mathrm{Tr}"] \arrow[d, hook] & F(S_l) \arrow[d] \\
& H_0(X'_0/S_0) \otimes F(X') \arrow[r, "\mathrm{Tr}"] & F(S_0)
\end{tikzcd}$$

By assumption, the element $(\Gamma_s - \Gamma_{s'}) \otimes \psi$ in the top left group goes to zero in $H_0(X'_0/S_0) \otimes F(X')$. Hence it vanishes in $H_0(X'/S_l) \otimes F(X')$ by the immersion of $H_0(X'/S)/n$ in $H_0(X'_0/S_0)/n$ of 7.19. Therefore $s^*(\psi) - s'^*(\psi)$ vanishes in $F(S_l)$. □

We conclude this lecture with a description of the behavior of the relative Picard group for finite morphisms. We will need this description in the proof of 20.9.

Definition 7.22. Let $(\bar{Y}, Y_\infty)$ and $(\bar{X}, X_\infty)$ be two good compactifications, say of Y and X, respectively. Any finite map $f : \bar{Y} \to \bar{X}$ which restricts to a map $f : Y \to X$, yields a map $f_* : \mathcal{O}^*(Y_\infty) \to \mathcal{O}^*(X_\infty)$ constructed as follows.

Consider $\alpha \in \mathcal{O}^*(Y_\infty)$. We may extend α to $\tilde{\alpha} \in \mathcal{O}^*(U)$ where U is an affine open neighborhood of Y_∞. Since f is finite, we may assume that $U = f^{-1}(V)$, where V is an open neighborhood of X_∞. Since V is normal, there is a norm map $N : \mathcal{O}^*(U) \to \mathcal{O}^*(V)$ (see 2.4). We define $f_*(\alpha) = N(\tilde{\alpha})|_{X_\infty}$. By 7.23 below, $f_*(\alpha)$ is independent of the choice of the extension $\tilde{\alpha}$.

Exercise 7.23. Let $f : U \to V$ be a finite morphism of normal schemes and let $Z \subset V$ be a reduced closed subscheme. If $\alpha \in \mathcal{O}^*(U)$ and $\alpha = 1$ on the reduced closed subscheme $f^{-1}(Z)$, show that $N(\alpha) = 1$ on Z.

Lemma 7.24. *Let $(\bar{Y}, Y_\infty)$ and $(\bar{X}, X_\infty)$ be good compactifications of Y and X, respectively. Let f be a finite map $f : \bar{Y} \to \bar{X}$ which restricts to a map $f : Y \to X$. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{O}^*(Y_\infty) \arrow[r] \arrow[d, "f_*"] & \mathrm{Pic}(\bar{Y}, Y_\infty) \arrow[r, "\cong"] & H_0(Y/S) \arrow[d] \\
\mathcal{O}^*(X_\infty) \arrow[r] & \mathrm{Pic}(\bar{X}, X_\infty) \arrow[r, "\cong"] & H_0(X/S),
\end{tikzcd}$$

where f_* was defined in 7.22 and the right vertical map is induced by the push-forward of cycles.

Proof. Choose $\alpha \in \mathcal{O}^*(Y_\infty)$ and extend it to a rational function t on $\bar{Y}$ which is regular in a neighborhood of the form $f^{-1}(V)$. By definition, $f_*(\alpha)$ extends to the regular function $N(t)$ on V. The horizontal maps send α and $f_*(\alpha)$ to $(\mathcal{O}_{\bar{Y}}, \alpha)$ and $(\mathcal{O}_{\bar{X}}, f_*\alpha)$. Let D and D' be the Weil divisors on $\bar{Y}$ and $\bar{X}$ associated to t and $N(t)$, respectively. We may regard D and D' as classes in $C_0(Y/S)$ and $C_0(X/S)$. By 7.15, D and D' represent the images of $(\mathcal{O}_{\bar{Y}}, \alpha)$ and $(\mathcal{O}_{\bar{X}}, f_*\alpha)$ in $H_0(Y/S)$ and $H_0(X/S)$, respectively. The right vertical map send D to D' because $D' = \mathrm{div}(Nt)$ is the push-forward of $D = \mathrm{div}(t)$ (see [Ful84, 1.4]). $\square$

Lecture 8

Derived tensor products

The goal of this lecture is to define a tensor product on the derived category of étale sheaves with transfer, starting with the tensor product $X \otimes Y = X \times Y$ on Cor_k defined in 1.9. For this we first need to build a total tensor product on the category $\mathbf{PST}(k)$, and this construction makes sense in somewhat greater generality.

Let $\mathcal{A}$ be a small additive category. We define $\mathbb{Z}(\mathcal{A})$ to be the category of all additive presheaves on $\mathcal{A}$, i.e., all contravariant additive functors $F : \mathcal{A} \to \mathbf{Ab}$. It is an abelian category. The Yoneda embedding $h : \mathcal{A} \to \mathbb{Z}(\mathcal{A})$ allows us to define the additive category $\mathcal{A}^{\oplus}$ as the closure of $\mathcal{A}$ under infinite direct sums in $\mathbb{Z}(\mathcal{A})$. If X_i are in $\mathcal{A}$, we will consider $X = \oplus X_i$ to be the object of $\mathcal{A}^{\oplus}$ corresponding to the presheaf $h_X = \oplus h_{X_i}$ in $\mathbb{Z}(\mathcal{A})$.

More generally, if R is a ring, we define $R(\mathcal{A})$ to be the (abelian) category of all additive functors $F : \mathcal{A} \to R$-**mod**. By abuse of notation, we will write h_X for the functor $A \mapsto R \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A, X)$ and call it "representable".

Lemma 8.1. *Every representable presheaf h_X is a projective object of $R(\mathcal{A})$, every F in $R(\mathcal{A})$ has a projective resolution by representable functors, and every projective object of $R(\mathcal{A})$ is a direct summand of a representable functor.*

Proof. Since $\operatorname{Hom}_{R(\mathcal{A})}(h_X, F) \cong F(X)$, each h_X is a projective object in $R(\mathcal{A})$. Moreover every F in $R(\mathcal{A})$ is a quotient of some h_X, $X \in \mathcal{A}^{\oplus}$, because of the natural surjection

$$\bigoplus_{X \text{ in } \mathcal{A}} \bigoplus_{\substack{x \in F(X) \\ x \neq 0}} h_X \xrightarrow{x} F \qquad \square$$

Now suppose that $\mathcal{A}$ has an additive symmetric monoidal structure $\otimes$, such as $\mathcal{A} = Cor_k$. (By this, we mean that $\otimes$ commutes with finite direct sums; see 8A.3.) We may extend $\otimes$ to a tensor product on $\mathcal{A}^{\oplus}$ in the obvious way, and this extends to tensor product of projectives. We now extend $\otimes$ to a tensor product on all of $R(\mathcal{A})$.

If F and G are in $R(\mathcal{A})$, we can form the presheaf tensor product $(F \otimes_R G)(X) = F(X) \otimes_R G(X)$. However, it does not belong to $R(\mathcal{A})$, since $F \otimes_R G$ is not additive. In order to get a tensor product on $R(\mathcal{A})$, we need a more complicated construction.

Our construction of $\otimes$ is dictated by the requirement that if X and Y are in $\mathcal{A}$, then the tensor product $h_X \otimes h_Y$ of their representable presheaves should be represented by $X \otimes Y$. As a first step, note that we can extend $\otimes$ to a tensor product $\otimes : \mathcal{A}^{\oplus} \times \mathcal{A}^{\oplus} \to \mathcal{A}^{\oplus}$ commuting with $\oplus$. Thus if L_1 and L_2 are in the category $\mathbf{Ch}^{-}(\mathcal{A}^{\oplus})$ of bounded above cochain complexes $(\cdots \to F^n \to 0 \to \cdots)$, the chain complex $L_1 \otimes L_2$ is defined as the total complex of the double complex $L_1^* \otimes L_2^*$.

Definition 8.2. If F and G are objects of $R(\mathcal{A})$, choose projective resolutions $P_* \to F$ and $Q_* \to G$ and define $F \otimes^{\mathbb{L}} G$ to be $P \otimes Q$, i.e., $\mathrm{Tot}(P_* \otimes Q_*)$. We define the tensor product and Hom presheaves to be:

$$F \otimes G = H_0(F \otimes^{\mathbb{L}} G)$$

$$\underline{Hom}(F, G) : X \mapsto \mathrm{Hom}_{R(\mathcal{A})}(F \otimes h_X, G)$$

Since any two projective resolutions of F are chain homotopy equivalent, the chain complex $F \otimes^{\mathbb{L}} G$ is well-defined up to chain homotopy equivalence, and similarly for $\underline{Hom}(F, G)$. In particular, since h_X and h_Y are projective, we have $h_X \otimes^{\mathbb{L}} h_Y = h_X \otimes h_Y = h_{X \otimes Y}$ for all X and Y in $\mathcal{A}^{\oplus}$.

The following result implies that $R(\mathcal{A})$ is an additive symmetric monoidal category (see 8A.3).

Lemma 8.3. *The functor $\underline{Hom}(F, -)$ is right adjoint to $F \otimes -$. In particular, $\underline{Hom}(F, -)$ is left exact and $F \otimes -$ is right exact.*

Proof. Because $R(\mathcal{A})$ has enough projectives, it suffices to observe that

$$\mathrm{Hom}_{R(\mathcal{A})}(h_X, \underline{Hom}(h_Y, G)) = G(X \otimes Y) = \mathrm{Hom}_{R(\mathcal{A})}(h_X \otimes h_Y, G). \qquad \square$$

Example 8.4. If $\mathcal{A}$ is the category of free R-modules over a commutative ring R, $R(\mathcal{A})$ is equivalent to the category of all R-modules; the presheaf associated to M is $M \otimes_R$, and $\underline{Hom}$ and $\otimes$ are the familiar Hom_R and $\otimes_R$.

Exercise 8.5. If F_i and G_i are in $R(\mathcal{A})$, show that there is a natural map

$$\underline{Hom}(F_1, G_1) \otimes \underline{Hom}(F_2, G_2) \to \underline{Hom}(F_1 \otimes F_2, G_1 \otimes G_2),$$

compatible with the monoidal pairing $\mathrm{Hom}_{\mathcal{A}}(U \times A_1, X_1) \otimes \mathrm{Hom}_{\mathcal{A}}(U \times A_2, X_2) \to \mathrm{Hom}_{\mathcal{A}}(U \times U \times A_1 \times A_2, X_1 \times X_2) \to \mathrm{Hom}_{\mathcal{A}}(U \times A_1 \times A_2, X_1 \times X_2)$.

Remark 8.6. If the (projective) objects h_X are flat, i.e., $h_X \otimes -$ is an exact functor, then $\otimes$ is called a balanced functor ([Wei94, 2.7.7]). In this case $F \otimes^{\mathbb{L}} G$ agrees (up to chain equivalence) with the usual left derived functor $\mathbb{L}(F \otimes -)G$. But we do not know when the h_X are flat. It is true in example 8.4, but probably not true in $\mathbf{PST} = \mathbb{Z}(Cor_k)$.

We can now extend $\otimes^{\mathbb{L}}$ to a total tensor product on the category $\mathbf{Ch}^- R(\mathcal{A})$ of bounded above cochain complexes $(\cdots \to F^n \to 0 \to \cdots)$. This would be the usual derived functor if $\otimes$ were balanced (see [Wei94, 10.6]), and our construction is parallel. If C is a complex in $\mathbf{Ch}^- R(\mathcal{A})$, there is a quasi-isomorphism $P \xrightarrow{\simeq} C$ with P a complex of projective objects. Any such complex P is called a projective resolution of C, and any other projective resolution of C is chain homotopic to P; see [Wei94, 5.7]. If D is any other complex in $\mathbf{Ch}^- R(\mathcal{A})$, and $Q \xrightarrow{\simeq} D$ is a projective resolution, we define

$$C \otimes^{\mathbb{L}} D = P \otimes Q.$$

Because P and Q are bounded above, each $(P \otimes Q)^n = \oplus_{i+j=n} P^i \otimes Q^j$ is a finite sum, and $C \otimes^{\mathbb{L}} D$ is bounded above. Because P and Q are defined up to chain homotopy, the complex $C \otimes^{\mathbb{L}} D$ is independent (up to chain homotopy equivalence) of the choice of P and Q. There is a natural map $C \otimes^{\mathbb{L}} D \to C \otimes D$, which extends the map $F \otimes^{\mathbb{L}} G \to F \otimes G$ of definition 8.2.

Lemma 8.7. *Let C, C' and D be bounded above complexes of presheaves.*

1. *If C and D are complexes over $\mathcal{A}^{\oplus}$, or complexes of projectives, then $C \otimes^{\mathbb{L}} D \xrightarrow{\simeq} C \otimes D$ is a chain homotopy equivalence.*

2. *If $f : C \xrightarrow{\simeq} C'$ is a quasi-isomorphism of complexes, then $C \otimes^{\mathbb{L}} D \to C' \otimes^{\mathbb{L}} D$ is a chain homotopy equivalence.*

Proof. If C is a complex over $\mathcal{A}^{\oplus}$, it is a complex of projectives. We may take $P = C$ in the definition of $\otimes^{\mathbb{L}}$: $C \otimes^{\mathbb{L}} D = C \otimes Q$. If D is also a complex of projectives, we may take $Q = D$ as well. Part 1 is now immediate. In part 2, we may take P to be a projective resolution of both C and C', so that $C \otimes^{\mathbb{L}} D = C' \otimes^{\mathbb{L}} D = P \otimes Q$. □

Proposition 8.8. *The derived category* $\mathbf{D}^{-}R(\mathcal{A})$*, equipped with* $\otimes^{\mathbb{L}}$*, is a tensor triangulated category.*

Proof. The category $\mathcal{P}$ of projective objects in $R(\mathcal{A})$ is additive symmetric monoidal, and $\mathbf{D}^{-}R(\mathcal{A})$ is equivalent to the chain homotopy category $\mathbf{K}^{-}(\mathcal{P})$ by [Wei94, 10.4.8]. By 8A.4, this is a tensor triangulated category under $\otimes$. The result now follows from the natural isomorphism $\otimes \cong \otimes^{\mathbb{L}}$ in $\mathcal{P}$ of 8.7. □

Definition 8.9. If C and D are bounded above complexes of presheaves, there is a canonical map from the presheaf tensor product $C \otimes_R D$ to the tensor product $C \otimes D$. By right exactness of $\otimes_R$ and $\otimes$ (see 8.3), it suffices to construct a natural map of presheaves $h_X \otimes_R h_Y \to h_{X \otimes Y}$. For U in $\mathcal{A}$, this is just the monoidal product in $\mathcal{A}$, followed by the diagonal $\Delta : U \to U \otimes U$:

$$\begin{aligned} h_X(U) \otimes_R h_Y(U) = \operatorname{Hom}_{\mathcal{A}}(U, X) \otimes_R \operatorname{Hom}_{\mathcal{A}}(U, Y) &\xrightarrow{\otimes} \\ \operatorname{Hom}_{\mathcal{A}}(U \otimes U, X \otimes Y) &\xrightarrow{\Delta^*} \operatorname{Hom}_{\mathcal{A}}(U, X \otimes Y) = h_{X \otimes Y}(U). \end{aligned}$$

Having disposed with these generalities, we now specialize to the case where $\mathcal{A}$ is Cor_k and $\otimes$ is the tensor product $X \otimes Y = X \times Y$ of 1.9. We have the Yoneda embedding

$$Cor_k \subset Cor_k^{\oplus} \subset \mathbf{PST}(k).$$

We will write $\otimes^{tr}$ for the tensor product on $\mathbf{PST} = \mathbb{Z}(Cor_k)$, or on $\mathbf{PST}(k, R) = R(Cor_k)$, and $\otimes_L^{tr}$ for $\otimes^{\mathbb{L}}$. Thus there are natural maps $C \otimes_L^{tr} D \to C \otimes^{tr} D$.

Example 8.10. By lemma 8.1, $h_X = R_{tr}(X)$ is projective and

$$R_{tr}(X) \otimes^{tr} R_{tr}(Y) = R_{tr}(X \times Y).$$

Similarly if (X_i, x_i) are pointed schemes then the $R_{tr}(X_i, x_i)$ are projective and from 2.12 we see that

$$R_{tr}(X_1, x_1) \otimes^{tr} \cdots \otimes^{tr} R_{tr}(X_n, x_n) = R_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n)).$$

In particular, $R_{tr}(\mathbb{G}_m)^{\otimes^{tr} n} = R_{tr}(\mathbb{G}_m^{\wedge n})$.

The next example, in which $R = \mathbb{Z}$, shows that $\otimes^{tr}$ does not behave well on locally constant sheaves.

Example 8.11. The complex $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ is a projective resolution of $\mathbb{Z}/n$, so we have $\mathbb{Z}/n \otimes^{tr} \mathbb{Z}_{tr}(X) = \mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}_{tr}(X) = (\mathbb{Z}/n)_{tr}(X)$ by 8.7.

If $\sqrt{-1} \notin k$ and $l = k(\sqrt{-1})$, let $\mathbb{Z}_\epsilon = \mathbb{Z}_{tr}(l)/\mathbb{Z}$ denote the locally constant sheaf corresponding to the sign representation of $G = Gal(l/k)$. We see from 8.7 that $\mathbb{Z}/n \otimes^{tr}_L \mathbb{Z}_\epsilon$ is quasi-isomorphic to the complex $(\mathbb{Z}/n) \otimes^{\mathbb{L}} (\mathbb{Z} \to \mathbb{Z}_{tr}(l))$, i.e.,

$$0 \to \mathbb{Z}/n \to (\mathbb{Z}/n)_{tr}(l) \to 0.$$

Hence the presheaf $(\mathbb{Z}/n) \otimes^{tr} \mathbb{Z}_\epsilon$ sends $\operatorname{Spec} k$ to 0 and $\operatorname{Spec} l$ to $\mathbb{Z}/n$. If $n = 4$, this is not an étale sheaf because $(\mathbb{Z}_\epsilon/4\mathbb{Z}_\epsilon)^G \neq 0$. It is easy to see, however, that its sheafification is the locally constant étale sheaf:

$$\big((\mathbb{Z}/4) \otimes^{tr} \mathbb{Z}_\epsilon\big)_{\acute{e}t} \cong \mu_4.$$

The étale sheaf μ_4 is the tensor product $(\mathbb{Z}/4) \otimes_{\acute{e}t} \mathbb{Z}_\epsilon$ of the two underlying étale sheaves.

Definition 8.12. If F and G are presheaves of R-modules with transfer, we write $F \otimes^{tr}_{\acute{e}t} G$ for $(F \otimes^{tr} G)_{\acute{e}t}$, the étale sheaf associated to $F \otimes^{tr} G$. If C and D are bounded above complexes of presheaves with transfer, we shall write $C \otimes^{tr}_{\acute{e}t} D$ for $(C \otimes^{tr} D)_{\acute{e}t}$, and $C \otimes^{tr}_{L,\acute{e}t} D$ for $(C \otimes^{tr}_L D)_{\acute{e}t} \simeq P \otimes^{tr}_{\acute{e}t} Q$, where P and Q are complexes of representable sheaves with transfers, and $P \simeq C$ and $Q \simeq D$. There is a natural map $C \otimes^{tr}_{L,\acute{e}t} D \to C \otimes^{tr}_{\acute{e}t} D$, induced by $C \otimes^{tr}_L \to C \otimes^{tr} D$.

Lemma 8.13. *If F, F' are étale sheaves of R-modules with transfer, and F is locally constant, then the map of 8.9 induces an isomorphism*

$$F \otimes_{\acute{e}t} F' \xrightarrow{\cong} F \otimes^{tr}_{\acute{e}t} F'.$$

Proof. Let F correspond to the discrete Galois module M. As $M = \cup M^H$ and $\otimes^{tr}$ commutes with colimits, we may assume that $M = M^H$ for some open normal H of $Gal = Gal(k_{sep}/k)$. Thus M is a G-module. Choose a presentation over $R[G]$:

$$\oplus R[G]^\alpha \to \oplus R[G]^\beta \to M \to 0.$$

As $\otimes_{ét}$ and $\otimes_{ét}^{tr}$ are both right exact, we may assume $M = R[G]$ and $F' = R_{tr}(X)$. If $L = (k_{sep})^H$ and $T = \mathrm{Spec}(L)$ then $F = R_{tr}(T)$ by exercise 6.10. But then $F \otimes^{tr} F' = R_{tr}(T \times X)$, so it suffices to observe that $R_{tr}(T) \otimes_{ét} R_{tr}(X) \to R_{tr}(T \times X)$ is an isomorphism. Since $T \times Y \to Y$ is an étale cover, it suffices to observe that for Y over T

$$\begin{aligned} R_{tr}(T) \otimes_{ét} R_{tr}(X)(Y) &\cong R[G] \otimes Cor(Y, X) \cong \\ &\cong R \otimes_{\mathbb{Z}} Cor(Y, T \times X) = R_{tr}(T \times X)(Y). \quad \square \end{aligned}$$

We are now going to show (in 8.16) that the tensor product $\otimes_{L,ét}^{tr}$ induces a tensor triangulated structure on the derived category of étale sheaves of R-modules with transfer. Using proposition 8.8, we have $C \otimes_{L,ét}^{tr} D \cong D \otimes_{L,ét}^{tr} C$, and it suffices to show that $\otimes_{L,ét}^{tr}$ preserves quasi-isomorphisms.

As a first step, fix Y and consider the right exact functor $\Phi(F) = R_{tr}(Y) \otimes_{ét}^{tr} F$, from the category $\mathbf{PST}(k, R)$ of presheaves of R-modules with transfer to the category of étale sheaves of R-modules with transfer. Its left derived functors $L_p \Phi(F)$ are the homology sheaves of the total left derived functor $R_{tr}(Y) \otimes_{L,ét}^{tr} F$. If C is a chain complex (bounded below in homological notation), the hyperhomology spectral sequence (see [Wei94, 5.7.6]) is

$$E_{p,q}^2 = L_p \Phi(H_q C) \Rightarrow \mathbb{L}_{p+q} \Phi(C).$$

Example 8.14. If $U \to X$ is an étale cover, consider the augmented Čech complex

$$\check{C} : \quad \ldots \to R_{tr}(U \times_X U) \to R_{tr}(U) \to R_{tr}(X) \to 0.$$

Since $\check{C}_{ét}$ is exact by 6.12, each homology presheaf $H_q(U/X) = H_q(\check{C})$ satisfies $H_q(U/X)_{ét} = 0$. By definition, $R_{tr}(Y) \otimes^{tr} \check{C}$ is the augmented Čech complex

$$\ldots \to R_{tr}(U \times_X U \times Y) \to R_{tr}(U \times Y) \to R_{tr}(X \times Y) \to 0$$

for the étale cover $U \times Y \to X \times Y$, so $R_{tr}(Y) \otimes_{ét}^{tr} \check{C}$ is again exact by 6.12. Thus $\mathbb{L}_n \Phi(\check{C}) = 0$ for all n. In particular, the 0^{th} homology presheaf $H_0(U/X)$ satisfies

$$\Phi\, H_0(U/X) = R_{tr}(Y) \otimes_{ét}^{tr} H_0(U/X) = H_0\left(R_{tr}(Y) \otimes_{ét}^{tr} \check{C}\right) = 0.$$

The following lemma shows that in fact every derived functor $L_n\,\Phi$ vanishes on $H_0(U/X)$.

Lemma 8.15. *Fix Y and set $\Phi = R_{tr}(Y)\otimes^{tr}_{\acute{e}t}$. If F is a presheaf of R-modules with transfer such that $F_{\acute{e}t} = 0$, then $L_n\,\Phi(F) = 0$ for all n.*

Proof. Suppose that $F_{\acute{e}t} = 0$. Each map $R_{tr}(X) \to F$ is defined by an $x \in F(X)$, and there is an étale cover $U_x \to X$ such that x vanishes in $F(U_x)$. Thus the composition $R_{tr}(U_x) \to R_{tr}(X) \to F$ is zero, i.e., the given map factors through the cokernel $H_0(U_x/X)$ of $R_{tr}(U_x) \to R_{tr}(X)$. It follows that the canonical surjection $\oplus_{X,x}R_{tr}(X) \to F$ factors through a surjection $\oplus_{X,x}H_0(U_x/X) \to F$. If K denotes the kernel of this surjection then $K_{\acute{e}t} = 0$.

We now proceed by induction on n, noting that $L_n\,\Phi = 0$ for $n < 0$. For $n = 0$, we know that $\Phi\,H_0(U_x/X) = 0$ by example 8.14. Since Φ is right exact, this yields $\Phi(F) = 0$. For $n > 0$, we may assume that the lemma holds for $L_p\,\Phi$ when $p < n$. From the exact sequence

$$\oplus_{X,x}(L_n\,\Phi)H_0(U_x/X) \to L_n\,\Phi(F) \to L_{n-1}\,\Phi(K)$$

we see that it suffices to prove that $(L_n\,\Phi)H_0(U/X) = 0$. We saw in 8.14 that $H_q(U/X)_{\acute{e}t} = 0$, so $L_p\,\Phi\,H_q(U/X) = 0$ by the inductive assumption. Hence the hypercohomology sequence for the complex $\check{C}$ collapses to yield

$$\mathbb{L}_n\,\Phi(\check{C}) \cong (L_n\,\Phi)H_0(\check{C}) = (L_n\,\Phi)H_0(U/X).$$

But we saw in example 8.14 that $L_n\,\Phi(\check{C}) = 0$, whence the result. ☐

Now we prove that $\otimes^{tr}_{L,\acute{e}t}$ preserves quasi-isomorphisms.

Proposition 8.16. *Let $f : C \to C'$ be a morphism of bounded above complexes of presheaves of R-modules with transfer. If f induces a quasi-isomorphism $C_{\acute{e}t} \to C'_{\acute{e}t}$ between the associated complexes of étale sheaves, then $C \otimes^{tr}_{L,\acute{e}t} D \to C' \otimes^{tr}_{L,\acute{e}t} D$ is a quasi-isomorphism for every D.*

Proof. If $P \xrightarrow{\simeq} C$ is a projective resolution of presheaves, then $P_{\acute{e}t} \to C_{\acute{e}t}$ is a quasi-isomorphism of complexes of étale sheaves. Thus we may assume that C, C' and D are complexes of representable presheaves. If A denotes the mapping cone of $C \to C'$, it suffices to show that $A \otimes^{tr}_{L,\acute{e}t} D = A \otimes^{tr}_{\acute{e}t} D$ is acyclic. As each row of the double complex underlying $A \otimes^{tr}_{\acute{e}t} D$ is a sum of terms $A \otimes^{tr}_{\acute{e}t} R_{tr}(Y)$, it suffices to show that $A \otimes^{tr}_{\acute{e}t} R_{tr}(Y)$ is acyclic. As in the proof of 8.15, its homology sheaves are the hyper-derived functors $\mathbb{L}_n\,\Phi(A)$, $\Phi = \otimes^{tr}_{ét} R_{tr}(Y)$. In the hypercohomology spectral sequence

$$E^2_{p,q} = L_p\,\Phi(H_q A) \Rightarrow \mathbb{L}_{p+q}\,\Phi(A)$$

the presheaves $H_q A$ have $(H_q A)_{ét} = 0$ because $A_{ét}$ is acyclic. By lemma 8.15 we have $L_q\,\Phi(H_q A) = 0$ for all p and q. Hence the spectral sequence collapses to yield $\mathbb{L}_n\,\Phi(A) = 0$ for all n, i.e., $\mathbb{L}\,\Phi(A) \simeq R_{tr}(Y) \otimes^{tr}_{ét} A$ is acyclic. □

Corollary 8.17. *The derived category of bounded above complexes of étale sheaves of R-modules with transfer is a tensor triangulated category.*

Proof. Combine 8.16 with 8.8 and 8A.7. □

Lemma 8.18. *Let F be a locally constant étale sheaf of flat R-modules. Then the map $E \otimes^{tr}_{L,ét} F \to E \otimes^{tr}_{ét} F$ is a quasi-isomorphism for every étale sheaf with transfers E.*

Proof. Suppose first that $E = R_{tr}(Y)$. Choose a resolution $C \to F$ in the category of locally constant sheaves in which each C_n is a sum of representables $R_{tr}(L_{n,\alpha})$ for finite Galois field extensions $L_{n,\alpha}$ of k. (This is equivalent to resolving the Galois module M corresponding to F by Galois modules $R[G_{n,\alpha}]$, and the existence of such a resolution of M is well known.) By proposition 8.16, $E \otimes^{tr}_{ét} C = E \otimes^{tr}_{L,ét} C$ is quasi-isomorphic to $E \otimes^{tr}_{L,ét} F$. By lemma 8.13,

$$E \otimes^{tr}_{ét} C = E \otimes_{ét} C \xrightarrow{\simeq} E \otimes_{ét} F \xleftarrow{\simeq} E \otimes^{tr}_{ét} F.$$

Hence the result is true for $E = R_{tr}(Y)$.

In the general case, choose a projective resolution $P \to E$ in the category of presheaves of R-modules with transfer. Then we have quasi-isomorphisms

$$E \otimes^{tr}_{L,ét} F = P \otimes^{tr}_{L,ét} F \xrightarrow{\simeq} P \otimes^{tr}_{ét} F \xrightarrow{\simeq} P \otimes_{ét} F.$$

Because sheafification is exact, $P \to E$ is also a resolution in the category of étale sheaves of R-modules. Since F is flat in this category, we have the final quasi-isomorphism:

$$P \otimes_{ét} F \xrightarrow{\simeq} E \otimes_{ét} F \xleftarrow{\simeq} E \otimes^{tr}_{ét} F.$$ □

It is clear that 8.18 also holds if E is a bounded above complex of étale sheaves with transfers.

Corollary 8.19. *In the derived category of étale sheaves of $\mathbb{Z}/m$-modules with transfer, the operation $M \mapsto M(1) = M \otimes^{tr}_{L,\acute{e}t} \mathbb{Z}/m(1)$ is invertible.*

Proof. Indeed, if μ_m^* is the Pontrjagin dual of μ_m, then combining 8.18, 8.13, and 4.8 yields:

$$\mu_m^* \otimes^{tr}_{L,\acute{e}t} \mathbb{Z}/m(1) \overset{8.18}{\simeq} \mu_m^* \otimes^{tr}_{\acute{e}t} \mathbb{Z}/m(1) \overset{8.13}{\cong} \mu_m^* \otimes_{\acute{e}t} \mathbb{Z}/m(1) \overset{4.8}{\cong} \mu_m^* \otimes_{\acute{e}t} \mu_m \cong \mathbb{Z}/m.$$

□

Exercise 8.20. If E and F are bounded above complexes of locally constant étale sheaves of R-modules, show that $E \otimes^{tr}_{L,\acute{e}t} F$ is quasi-isomorphic to $E \otimes^{\mathbb{L}}_R F$, their total tensor product as complexes of étale sheaves of R-modules. (Hint: Use 8.13, 8.16, and 8.18.)

Appendix 8A - Tensor Triangulated Categories

The notion of a tensor triangulated category is a generalization of the tensor product structure on the derived category of modules over a scheme, which played a central role in the development of the subject.

Definition 8A.1. A **tensor triangulated category** is an additive category with two structures: that of a triangulated category and that of a symmetric monoidal category. In addition, we are given natural isomorphisms r and l of the form

$$C[1] \otimes D \xrightarrow[l_{C,D}]{\cong} (C \otimes D)[1] \xleftarrow[r_{C,D}]{\cong} C \otimes D[1],$$

which commute in the obvious sense with the associativity, commutativity and unity isomorphisms. There are two additional axioms:

(TTC1) For any distinguished triangle $C_0 \longrightarrow C_1 \longrightarrow C_2 \xrightarrow{\partial} C_0[1]$ and any D, the following triangles are distinguished:

$$C_0 \otimes D \longrightarrow C_1 \otimes D \longrightarrow C_2 \otimes D \xrightarrow{l(\partial \otimes D)} (C_0 \otimes D)[1]$$

$$D \otimes C_0 \longrightarrow D \otimes C_1 \longrightarrow D \otimes C_2 \xrightarrow{r(D \otimes \partial)} (D \otimes C_0)[1].$$

(TTC2) For any C and D, the following diagram commutes up to multiplication by -1, i.e., $rl = -lr$:

$$\begin{tikzcd}
C[1] \otimes D[1] \arrow[r, "r"] \arrow[d, "l"'] \arrow[dr, phantom, "-1"] & (C[1] \otimes D)[1] \arrow[d, "l"] \\
(C \otimes D[1])[1] \arrow[r, "r"] & (C \otimes D)[2].
\end{tikzcd}$$

This description is not minimal. For example the commutativity isomorphism $\tau : C \otimes D \cong D \otimes C$ allows us to recover r from l and vice versa using the formula $\tau l \tau = r$. In addition, $l_{C,D}$ can be recovered from $l_{1,D} : \mathbf{1}[1] \otimes D \cong D[1]$. Moreover, if either of the two triangles in (TTC1) is distinguished, then both are distinguished.

The definition of tensor triangulated category that we have given is sufficient for our purposes. However, it is possible to add extra axioms in order to work with a richer structure. For example, many more axioms are postulated by May in [May01].

Exercise 8A.2. Show that the canonical isomorphisms $l^i r^j, r^j l^i : C[i] \otimes D[j] \cong (C \otimes D)[i+j]$ differ by $(-1)^{ij}$, and are interchanged by the twist isomorphism τ on $C \otimes D$ and $C[i] \otimes D[j]$.

Definition 8A.3. Let $\mathcal{A}$ be an additive category with a symmetric monoidal structure $\otimes$. We say that $\mathcal{A}$ is an **additive symmetric monoidal category** if $(\coprod A_i) \otimes B \cong \coprod (A_i \otimes B)$ for every finite direct sum $\coprod A_i$ in $\mathcal{A}$.

If C and D are bounded above complexes in $\mathcal{A}$, the tensor product $C \otimes D$ has $(C \otimes D)^n = \oplus_{p+q=n} C^p \otimes D^q$ and differential $d \otimes 1 + (-1)^p \otimes d$ on $C^p \otimes D^q$. It is associative.

We define the twist isomorphism $\tau : C \otimes D \to D \otimes C$ componentwise, as $(-1)^{pq}$ times the natural isomorphism $C^p \otimes D^q \to D^q \otimes C^p$ in $\mathcal{A}$. It is a straightforward exercise to verify that the category $\mathbf{Ch}^-(\mathcal{A})$ is an additive symmetric monoidal category.

The degree n part of each of $C \otimes D[1]$, $(C \otimes D)[1]$, and $C[1] \otimes D$ are the same, and we define $l_{C,D}$ to be the canonical isomorphism. The map $r_{C,D}$ is multiplication by $(-1)^p$ on the summand $C^p \otimes D^q$. A routine calculation verifies the following.

Proposition 8A.4. *Let $\mathcal{A}$ be an additive symmetric monoidal category. Then the chain homotopy category $\mathbf{K}^-(\mathcal{A})$ of bounded above cochain complexes is a tensor triangulated category.*

Example 8A.5. (See [Ver96].) Let $\mathcal{A}$ be the category of modules over a commutative ring, or more generally over a scheme. Then not only is $\mathbf{K}^-(\mathcal{A})$ a tensor triangulated category, but the total tensor product $\otimes^{\mathbb{L}}$ makes the derived category $\mathbf{D}^-(\mathcal{A})$ into a tensor triangulated category. In effect, $\mathbf{D}^-(\mathcal{A})$ is equivalent to the tensor triangulated subcategory of flat complexes in $\mathbf{K}^-(\mathcal{A})$.

Example 8A.6. The smash product of based topological spaces leads to another example. If $A \to X \to X/A \to SA$ is a cofibration sequence, there is a natural homeomorphism $(X/A) \wedge Y \cong (X \wedge Y)/(A \wedge Y)$; see [Whi78, III.2.3]. The suspension $SX = S^1 \wedge X$ has homeomorphisms

$$X \wedge (SY) \xrightarrow[r]{\cong} S(X \wedge Y) \xleftarrow[l]{\cong} (SX) \wedge Y$$

satisfying (TTC1) and (TTC2) up to homotopy. It follows easily that the stable homotopy category, which is triangulated by [Wei94, 10.9.18] and a symmetric monoidal category by [Ada74, III.4], is a tensor triangulated category.

If W is a saturated multiplicative system of morphisms in a triangulated category $\mathbf{D}$, closed under $\oplus$, translations, and cones, Verdier proved in [Ver96] that the localization $\mathbf{D}[W^{-1}]$ is also a triangulated category.

Proposition 8A.7. *Let $\mathbf{D}$ be a tensor triangulated category. Suppose that if $C \to C'$ is in W then $C \otimes D \to C' \otimes D$ is in W for every D in $\mathbf{D}$. Then the localization $\mathbf{D}[W^{-1}]$ is also a tensor triangulated category.*

Proof. Because each $\otimes D : \mathbf{D} \to \mathbf{D}$ preserves W, $\otimes$ induces a symmetric monoidal pairing $\mathbf{D}[W^{-1}] \times \mathbf{D}[W^{-1}] \to \mathbf{D}[W^{-1}]$ by the universal property of localization (applied to $W \times W$). Similarly, the natural isomorphisms r and l descend to $\mathbf{D}[W^{-1}]$. Axiom (TTC2) is automatic, and axiom (TTC1) may be routinely verified for Verdier's description of distinguished triangles in $\mathbf{D}[W^{-1}]$. $\square$

Exercise 8A.8. Let T be an invertible object in a symmetric monoidal category $\mathcal{C}$, i.e., an object such that $T \otimes U \cong \mathbf{1}$ for some U. It is well known that endomorphisms of $\mathbf{1}$ commute; show that the same must be true for endomorphisms of T. Then show that the cyclic permutation of $T \otimes (T \otimes T)$ must equal the identity morphism.

Let T be an object in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$. Let $\mathcal{C}[T^{-1}]$ denote the category whose objects are pairs (X, m) with X in $\mathcal{C}$ and $m \in \mathbb{Z}$; morphisms $(X, m) \to (Y, n)$ in $\mathcal{C}[T^{-1}]$ are just elements of the direct limit $\lim_{i\to\infty} \operatorname{Hom}(X \otimes T^{\otimes m+i}, Y \otimes T^{\otimes n+i})$, where the bonding maps are given by the functor $\otimes T : \mathcal{C} \to \mathcal{C}$. Composition is defined in the obvious way, and it's easy to check that $\mathcal{C}[T^{-1}]$ is a category. There is a universal functor $\mathcal{C} \to \mathcal{C}[T^{-1}]$ sending X to $(X, 0)$. Note that $(X, m) \cong X \otimes T^{\otimes m}$ in $\mathcal{C}[T^{-1}]$ for $m \geq 0$.

Exercise 8A.9. Let T be an object in a tensor triangulated category $\mathcal{C}$. Show that $\mathcal{C}[T^{-1}]$ is a triangulated category, and that $\mathcal{C} \to \mathcal{C}[T^{-1}]$ is triangulated.

In order for the formula $(X, m) \otimes (Y, n) = (X \otimes Y, m + n)$ to extend to a bifunctor on $\mathcal{C}[T^{-1}]$, we need to define the tensor $f \otimes g$ of two $\mathcal{C}[T^{-1}]$-morphisms in a natural way. In general, $\mathcal{C}$ need not be symmetric monoidal, as exercise 8A.8 above shows.

Proposition 8A.10. *Let T be an object in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ such that the cyclic permutation on $T^{\otimes 3}$ is the identity in $\mathcal{C}[T^{-1}]$. Then $(\mathcal{C}[T^{-1}], \otimes, \mathbf{1})$ is also a symmetric monoidal category.*

Proof. The hypothesis implies that permutations on $T^{\otimes n}$ commute with each other for $n \geq 3$. The many ways to define $f \otimes g$ on $X \otimes T^{m+i} \otimes Y \otimes T^{n+j}$ are indexed by the (i, j)-shuffles, and differ only by a permutation, so $f \otimes g$ is independent of this choice. Therefore the tensor product is a bifunctor on $\mathcal{C}[T^{-1}]$. The symmetric monoidal axioms may now be routinely verified as in [Ada74, III.4]. The hexagonal axiom, that the two isomorphisms from $X \otimes (Y \otimes Z)$ to $(Z \otimes X) \otimes Y)$ agree, follows because the cyclic permutation on $T^{\otimes 3}$ is the identity. $\square$

Corollary 8A.11. *Let T be an object in a tensor triangulated category $\mathcal{C}$ such that the cyclic permutation on $T^{\otimes 3}$ is the identity in $\mathcal{C}[T^{-1}]$. Then $\mathcal{C}[T^{-1}]$ is a tensor triangulated category.*

Proof. By 8A.9 and 8A.10, $\mathcal{C}[T^{-1}]$ is both triangulated and symmetric monoidal. The verification of the remaining axioms is straightforward. $\square$

Exercise 8A.12. Let T be an object in a tensor triangulated category $\mathbf{D}$ such that $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X \otimes T, Y \otimes T)$ is an isomorphism for every X and Y in $\mathbf{D}$. Show that $\mathbf{D}[T^{-1}]$ is a tensor triangulated category.

Lecture 9

$\mathbb{A}^1$-weak equivalence

In this section we define the notion of $\mathbb{A}^1$-weak equivalence between bounded above cochain complexes of étale sheaves with transfers, and $\mathbb{A}^1$-local complexes. The category $\mathbf{DM}^{\text{eff},-}_{ét}$ is obtained by inverting $\mathbb{A}^1$-weak equivalences. The main result in this lecture (9.32) is that when we restrict to sheaves of $\mathbb{Z}/n$-modules the category $\mathbf{DM}^{\text{eff},-}_{ét}$ is equivalent to the derived category of discrete Galois modules for the group $Gal(k_{sep}/k)$. We will use these ideas in the next lecture to identify étale motivic cohomology with ordinary étale cohomology.

Since quasi-isomorphic complexes will be $\mathbb{A}^1$-weak equivalent, it is appropriate to define the notion in the derived category $\mathbf{D}^- = \mathbf{D}^-(Sh_{ét}(Cor_k, R))$ of étale sheaves of R-modules with transfer. In $\mathbf{D}^-$, we have the usual shift, and

$$A \xrightarrow{f} B \longrightarrow \text{cone}(f) \longrightarrow A[1]$$

is a distinguished triangle for each map f. We refer the reader to [GM88] or [Wei94] for basic facts about derived categories. We will also need the notion of a thick subcategory, which was introduced by Verdier in [Ver96]. We will use Rickard's definition (see [Ric89]); this is slightly different from, but equivalent to, Verdier's definition.

Definition 9.1. A full additive subcategory $\mathcal{E}$ of $\mathbf{D}^-$ is **thick** if:

1. Let $A \to B \to C \to A[1]$ be a distinguished triangle. Then if two out of A, B, C are in $\mathcal{E}$ then so is the third.

2. if $A \oplus B$ is in $\mathcal{E}$ then both A and B are in $\mathcal{E}$.

If $\mathcal{E}$ is a thick subcategory of $\mathbf{D}^-$, we can form a quotient triangulated category $\mathbf{D}^-/\mathcal{E}$ as follows (see [Ver96]). Let $W_{\mathcal{E}}$ be the set of maps whose cone is in $\mathcal{E}$; $W_{\mathcal{E}}$ is a saturated multiplicative system of morphisms. Then $\mathbf{D}^-/\mathcal{E}$ is the localization $\mathbf{D}^-[W_{\mathcal{E}}^{-1}]$, which may be constructed using calculus of fractions; see [Wei94, 10.3.7]. In particular, a morphism $f : C \to C'$ becomes an isomorphism in $\mathbf{D}^-[W_{\mathcal{E}}^{-1}]$ if and only if f is in $W_{\mathcal{E}}$.

Definition 9.2. A morphism f in $\mathbf{D}^-$ is called an $\mathbb{A}^1$**-weak equivalence** if f is in $W_{\mathbb{A}} = W_{\mathcal{E}_{\mathbb{A}}}$, where $\mathcal{E}_{\mathbb{A}}$ is the smallest thick subcategory so that:

1. the cone of $R_{tr}(X \times \mathbb{A}^1) \to R_{tr}(X)$ is in $\mathcal{E}_{\mathbb{A}}$ for every smooth scheme X;

2. $\mathcal{E}_{\mathbb{A}}$ is closed under any direct sum that exists in $\mathbf{D}^-$.

We set $\mathbf{DM}_{\acute{e}t}^{\mathrm{eff},-}(k, R) = \mathbf{D}^-[W_{\mathbb{A}}^{-1}]$.

It is clear that the notion of $\mathbb{A}^1$-weak equivalence in $\mathbf{D}^- = \mathbf{D}^-(Sh(Cor_k, R))$ makes sense for other topologies. For the Nisnevich topology, the localization $\mathbf{DM}_{Nis}^{\mathrm{eff},-}(k, R)$ of $\mathbf{D}^-$ is the triangulated category of motivic complexes introduced and studied in [TriCa].

Lemma 9.3. *The smallest class in $\mathbf{D}^-$ which contains all the $R_{tr}(X)$ and is closed under quasi-isomorphisms, direct sums, shifts, and cones is all of $\mathbf{D}^-$.*

Proof. First we show that for any complex D_*, if all D_n are in the class, then so is D_*. If $\beta_n D$ is the brutal truncation $0 \to D_n \to D_{n-1} \to \cdots$ of D_*, then D_* is the union of the $\beta_n D$. Each $\beta_n D$ is a finite complex, belonging to the class, as an inductive argument shows. Since there is an exact sequence

$$0 \longrightarrow \oplus\, \beta_n D \longrightarrow \oplus\, \beta_n D \longrightarrow D_* \longrightarrow 0,$$

it follows that D_* is in the class.

Thus it suffices to show that each sheaf F is in the class. Now there is a resolution $L_* \to F$ by sums of the representable sheaves $R_{tr}(X)$, given by lemma 8.1. Since each L_n is in this class, so is L_* and hence F. □

Lemma 9.4. *If $f : C \to C'$ is an $\mathbb{A}^1$-weak equivalence, then for every D the map $f \otimes Id : C \otimes_{L,\acute{e}t}^{tr} D \to C' \otimes_{L,\acute{e}t}^{tr} D$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. Since $\otimes^{tr}_{L,\acute{e}t}$ commutes with cones and f is an $\mathbb{A}^1$-weak equivalence if and only if its cone is in $\mathcal{E}_{\mathbb{A}}$, it suffices to show that if C is in $\mathcal{E}_{\mathbb{A}}$, then $C \otimes^{tr}_{L,\acute{e}t} D$ is in $\mathcal{E}_{\mathbb{A}}$ for any D.

If $D = R_{tr}(X)$, consider the subcategory $\mathcal{E}$ of all C in $\mathbf{D}^-$ such that $C \otimes^{tr}_{L,\acute{e}t} D$ is in $\mathcal{E}_{\mathbb{A}}$. $\mathcal{E}$ is closed under direct sums and it is thick. Moreover, if Y is a smooth scheme, then $\mathcal{E}$ contains the cone of $R_{tr}(Y \times \mathbb{A}^1) \to R_{tr}(Y)$. Therefore $\mathcal{E}_{\mathbb{A}} \subseteq \mathcal{E}$.

Now fix C in $\mathcal{E}_{\mathbb{A}}$ and consider the full subcategory $\mathcal{D}$ of all D in $\mathbf{D}^-$ such that $C \otimes^{tr}_{L,\acute{e}t} D$ is in $\mathcal{E}_{\mathbb{A}}$. $\mathcal{D}$ is closed under direct sums, it is thick and we have seen that it contains $R_{tr}(X)$ for all X. By 9.3, we conclude that $\mathcal{D} = \mathbf{D}^-$. □

Corollary 9.5. *The product $\otimes^{tr}_{L,\acute{e}t}$ endows $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, R)$ with the structure of a tensor triangulated category.*

Proof. Given 8.17, this follows from 9.4 and proposition 8A.7. □

Remark 9.6. The category $\mathbf{DM}^{-}_{\acute{e}t}(k, R)$ is obtained from $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, R)$ by inverting the Tate twist operation $M \mapsto M(1) = M \otimes^{tr}_{L,\acute{e}t} R(1)$. If $R = \mathbb{Z}/m$, then the Tate twist is already invertible by 8.19, so we have

$$\mathbf{DM}^{-}_{\acute{e}t}(k, \mathbb{Z}/m) = \mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, \mathbb{Z}/m).$$

For any coefficients R, it will follow from 8A.11 and 15.8 below that $\mathbf{DM}^{-}_{\acute{e}t}(k, R)$ is always a tensor triangulated category.

Definition 9.7. Two morphisms $F \overset{f}{\underset{g}{\rightrightarrows}} G$ of sheaves of R-modules with transfer are called $\mathbb{A}^1$**-homotopic** if there is a map $h : F \otimes^{tr} R_{tr}(\mathbb{A}^1) \to G$ so that the restrictions of h along $R \overset{1}{\underset{0}{\rightrightarrows}} R_{tr}(\mathbb{A}^1)$ coincide with f and g.

If G is an étale sheaf, h factors through (and is determined by) a map $F_{\acute{e}t} \otimes^{tr}_{L,\acute{e}t} R_{tr}(\mathbb{A}^1) \to G$.

Example 9.8. Suppose we are given two maps $f, g : X \to Y$ such that the induced maps $\mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(Y)$ are $\mathbb{A}^1$-homotopic in the sense of 9.7. By the Yoneda lemma, this is equivalent to saying that f and g are restrictions of some $h \in Cor(X \times \mathbb{A}^1, Y)$, i.e., that f and g are $\mathbb{A}^1$-homotopic maps in the sense of 2.24.

Lemma 9.9. *Let $f, g : F \to G$ be two maps between étale sheaves with transfers. If f and g are $\mathbb{A}^1$-homotopic, then $f = g$ in $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, R)$.*

Proof. Any two sections of $\mathbb{A}^1 \to \operatorname{Spec} k$ yield the same map $R \to R_{tr}(\mathbb{A}^1)$ in the localized category $\mathbf{DM}^{\text{eff},-}_{\acute{e}t}(k,R)$, namely the inverse of the $\mathbb{A}^1$-weak equivalence $R_{tr}(\mathbb{A}^1) \to R$. Therefore the maps:

$$F \underset{F\times 1}{\overset{F\times 0}{\rightrightarrows}} R_{tr}(\mathbb{A}^1) \otimes^{tr}_{L,\acute{e}t} F \xrightarrow{h} G$$

are the same in the localized category. ∎

There is a mistake in the proof of the corresponding Lemma 3.2.5 in [TriCa] as the proof there assumes that $\mathbb{Z}_{tr}(\mathbb{A}^1)$ is flat in Cor_k. If we replace $\otimes$ by $\otimes_L$ in loc. cit., the proof goes through as written.

Corollary 9.10. *Every $\mathbb{A}^1$-homotopy equivalence is an $\mathbb{A}^1$-weak equivalence.*

Our next goal is to show that, $F \to C_*F$ is always an $\mathbb{A}^1$-weak equivalence (see 9.14 below). Hence $F \cong C_*F$ in $\mathbf{DM}^{\text{eff},-}_{\acute{e}t}(k,R)$.

Lemma 9.11. *Let $B \to B'$ be a map of double complexes which are vertically bounded above in the sense that there is a Q so that $B^{*,q} = (B')^{*,q} = 0$ for all $q \geq Q$. Suppose that all rows are weak equivalences and that* $\operatorname{Tot}(B)$ *and* $\operatorname{Tot}(B')$ *are bounded above.*

Then $\operatorname{Tot}(B) \to \operatorname{Tot}(B')$ *is an $\mathbb{A}^1$-weak equivalence.*

Proof. Let $S(n)$ be the double subcomplex of B consisting of the B^{pq} for $q \geq n$. Then $\operatorname{Tot} S(n+1)$ is a subcomplex of $\operatorname{Tot} S(n)$ whose cokernel is a shift of the n-th row of B. If $S'(n)$ is defined similarly, then each $\operatorname{Tot} S(n) \to \operatorname{Tot} S'(n)$ is an $\mathbb{A}^1$-weak equivalence by induction on n. Now $Sh_{\acute{e}t}(Cor_k, R)$ satisfies (AB4), meaning that $\oplus$, and hence Tot, is exact. Hence there is a short exact sequence of complexes

$$0 \longrightarrow \bigoplus_{n=1}^{\infty} \operatorname{Tot} S(n) \xrightarrow{id-\text{shift}} \bigoplus_{n=1}^{\infty} \operatorname{Tot} S(n) \longrightarrow \operatorname{Tot} B \longrightarrow 0$$

and similarly for B'. Since $\oplus \operatorname{Tot} S(n) \to \oplus \operatorname{Tot} S'(n)$ is an $\mathbb{A}^1$-weak equivalence, so is $\operatorname{Tot} B \to \operatorname{Tot} B'$. ∎

Corollary 9.12. *If $f : C \to C'$ is a morphism of bounded above complexes, and $f_n : C_n \to C'_n$ is in $W_{\mathbb{A}}$ for every n, then f is in $W_{\mathbb{A}}$.*

Proof. This is a special case of 9.11. ∎

Lemma 9.13. *For every F and every n, the map $F \xrightarrow{s} \underline{Hom}(R_{tr}(\Delta^n), F) = C_n(F)$ is an $\mathbb{A}^1$-homotopy equivalence. A fortiori, it is an $\mathbb{A}^1$-weak equivalence.*

Proof. Since Δ^n is isomorphic to $\mathbb{A}^n$ as a scheme, we have $C_n(F) \cong C_1 C_{n-1}(F)$. Thus we may suppose that $n = 1$. We define a map $m : C_1 F \to C_2 F$ as follows. For each X, the map

$$m_X : C_1(F)(X) = F(X \times \mathbb{A}^1) \to F(X \times \mathbb{A}^2) = C_2(F)$$

is induced by the multiplication map $\mathbb{A}^2 \to \mathbb{A}^1$ by crossing it with X and applying F. Since $C_2 F = \underline{Hom}(R_{tr}(\mathbb{A}^1), C_1 F)$, the adjunction of 8.2 associates to m a map $h : C_1 F \otimes^{tr} R_{tr}(\mathbb{A}^1) \to C_1 F$. Similarly the inclusions $\mathbb{A}^1 \times \{i\} \subset \mathbb{A}^2$ induce maps $\eta_i : C_2 F \to C_1 F$, and the compositions $\eta_i m : C_1 F \to C_1 F$ are adjoint to the restriction of h along $i : R \to R_{tr}(\mathbb{A}^1)$. Hence h induces an $\mathbb{A}^1$-homotopy between the identity $(\eta_1 m)$ and the composite

$$C_1 F \xrightarrow{\partial_0} F \xrightarrow{s} C_1 F,$$

corresponding to $\eta_0 m$. Since $\partial_0 s$ is the identity on F, s and ∂_0 are inverse $\mathbb{A}^1$-homotopy equivalences. They are $\mathbb{A}^1$-weak equivalences by 9.10. □

Lemma 9.14. *For every bounded above complex F of sheaves of R-modules with transfer, the morphism $F \to C_*(F)$ is an $\mathbb{A}^1$-weak equivalence. Hence $F \cong C_*(F)$ in $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, R)$.*

Proof. By 9.11, we may assume that F is a sheaf. Consider the diagram whose rows are chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F \arrow[d, "="] \\
\cdots \arrow[r, "0"] & F \arrow[r, "1"] \arrow[d, "\simeq_{\mathbb{A}^1}"] & F \arrow[r, "0"] \arrow[d, "\simeq_{\mathbb{A}^1}"] & F \arrow[d, "\simeq_{\mathbb{A}^1}"] \\
\cdots \arrow[r] & C_2 F \arrow[r] & C_1 F \arrow[r] & F.
\end{tikzcd}$$

The first two rows are quasi-isomorphic. Now $F \simeq_{\mathbb{A}^1} C_n(F)$ by 9.13. Using 9.12, we see that the second and third rows are $\mathbb{A}^1$-weak equivalent. □

Example 9.15. The identity map on $\mathcal{O}$ is $\mathbb{A}^1$-homotopic to zero by 2.22 and 9.14. Hence $\mathcal{O}$ is isomorphic to zero in $\mathbf{DM}_{\acute{e}t}^{\mathrm{eff},-}(k)$. When $\mathrm{char}\, k = \ell > 0$ the Artin-Schrier sequence of étale sheaves [Mil80, II 2.18(c)]

$$0 \longrightarrow \mathbb{Z}/\ell \longrightarrow \mathcal{O} \xrightarrow{1-\phi} \mathcal{O} \longrightarrow 0$$

shows that $\mathbb{Z}/\ell \cong 0$ in $\mathbf{DM}_{\acute{e}t}^{\mathrm{eff},-}(k)$. Here R may be either $\mathbb{Z}$ or $\mathbb{Z}/\ell$.

Definition 9.16. An object L in $\mathbf{D}^-$ is called $\mathbb{A}^1$**-local** if for all $\mathbb{A}^1$-weak equivalences $K \to K'$ the induced map $\mathrm{Hom}(K', L) \to \mathrm{Hom}(K, L)$ is bijective.

Lemma 9.17. *An object L in $\mathbf{D}^-$ is $\mathbb{A}^1$-local if and only if $\mathrm{Hom}(R_{tr}(X)[n], L) \to \mathrm{Hom}(R_{tr}(X \times \mathbb{A}^1)[n], L)$ is an isomorphism for all X and n.*

Proof. Let $\mathcal{K}$ be the full subcategory of all K for which $\mathrm{Hom}(K[n], L) = 0$ for all n. Clearly, $\mathcal{K}$ is a thick subcategory of $\mathbf{D}^-$ and it is closed under direct sums and shifts. Under the given hypothesis, $\mathcal{K}$ contains the cone of every map $R_{tr}(X \times \mathbb{A}^1) \to R_{tr}(X)$. By definition, $\mathcal{E}_{\mathbb{A}}$ is a subcategory of $\mathcal{K}$, i.e., L is $\mathbb{A}^1$-local. □

Lemma 9.18. *If $f : K \to K'$ is an $\mathbb{A}^1$-weak equivalence and K, K' are $\mathbb{A}^1$-local then f is an isomorphism in $\mathbf{D}^-$, i.e., a quasi-isomorphism of complexes of étale sheaves with transfers.*

Proof. By definition, f induces bijections $\mathrm{Hom}(K', K) \cong \mathrm{Hom}(K, K)$ and $\mathrm{Hom}(K', K') \cong \mathrm{Hom}(K, K')$. Hence there is a unique $g : K' \to K$ so that $fg = 1_K$, and $f(gf) = (fg)f = f$ implies that $gf = 1_{K'}$. □

Corollary 9.19. *If F is $\mathbb{A}^1$-local then $F \cong C_*F$ in $\mathbf{D}^-$.*

Lemma 9.20. *If Y is $\mathbb{A}^1$-local then for every X in $\mathbf{D}^-$*

$$\mathrm{Hom}_{\mathbf{DM}_{\acute{e}t}^{\mathrm{eff},-}(k,R)}(X, Y) = \mathrm{Hom}_{\mathbf{D}^-}(X, Y).$$

Proof. By the calculus of fractions [Wei94, 10.3.7], the left side consists of equivalence classes of diagrams $X \xleftarrow{s} K \longrightarrow Y$ with s in $W_{\mathbb{A}}$. It suffices to show that if $K \to K'$ is an $\mathbb{A}^1$-weak equivalence then $\mathrm{Hom}(K', Y) = \mathrm{Hom}(K, Y)$. But this holds since Y is $\mathbb{A}^1$-local. □

Definition 9.21. An étale sheaf with transfers F is **strictly $\mathbb{A}^1$-homotopy invariant** if the map $H^n_{\acute{e}t}(X, F) \to H^n_{\acute{e}t}(X \times \mathbb{A}^1, F)$ is bijective for all smooth X and every $n \in \mathbb{N}$. In particular for $n = 0$ we must have that F is homotopy invariant (2.14).

Lemma 9.22. *([SGA4, XV 2.2]) If R is of torsion prime to* $\operatorname{char} k$ *then any locally constant sheaf of R-modules is strictly $\mathbb{A}^1$-homotopy invariant.*

Lemma 9.23. *Let F be an étale sheaf of R-modules with transfers. Then F is $\mathbb{A}^1$-local if and only if F is strictly $\mathbb{A}^1$-homotopy invariant.*

Proof. By 6.23 or 6.24, we have

$$\operatorname{Hom}_{\mathbf{D}^-}(R_{tr}(X), F[i]) = \operatorname{Ext}^{\,i}_{Sh_{\acute{e}t}(Cor_k, R)}(R_{tr}(X), F) = H^i_{\acute{e}t}(X, F)$$

for every smooth X. Since $R_{tr}(X \times \mathbb{A}^1)[n] \to R_{tr}(X)[n]$ is an $\mathbb{A}^1$-weak equivalence for all n, 9.17 shows that F is $\mathbb{A}^1$-local if and only if the induced map

$$H^{-n}_{\acute{e}t}(X, F) = \operatorname{Hom}(R_{tr}(X)[n], F) \to \operatorname{Hom}(R_{tr}(X \times \mathbb{A}^1)[n], F) = H^{-n}_{\acute{e}t}(X \times \mathbb{A}^1, F)$$

is an isomorphism, that is, if and only if F is strictly $\mathbb{A}^1$-homotopy invariant. $\square$

Here is a special case of 9.23 which includes the sheaves $\mu_n^{\otimes q}$. It follows by combining 9.22 with 9.23.

Corollary 9.24. *Let M be a locally constant étale sheaf of torsion prime to* $\operatorname{char} k$. *Then M is $\mathbb{A}^1$-local.*

We now make the running assumption that R is a commutative ring and that $cd_R(k) < \infty$, i.e., k is a field having finite étale cohomological dimension for coefficients in R. This assumption allows us to invoke a classic result from [SGA4].

Lemma 9.25. *([SGA4], [Mil80]) Let X be a scheme of finite type over k. If k has finite R-cohomological dimension d then $cd_R(X) \le d + 2\dim_k X$.*

Corollary 9.26. $\operatorname{Ext}^n(R_{tr}(X), F) = 0$ *when $n \gg 0$.*

Proof. $\operatorname{Ext}^n(R_{tr}(X), F) \cong H^n_{\acute{e}t}(X, F)$ by 6.24. $\square$

If C is a chain complex of sheaves, each cohomology $H^n(C)$ is a presheaf. We write $a_{ét}H^n(C)$ for its associated sheaf.

Lemma 9.27. *For every (bounded above) chain complex C there is a bounded, convergent spectral sequence:*

$$E_2^{p,q} = \mathrm{Ext}^p(R_{tr}(X), a_{ét}H^q(C)) \Longrightarrow \mathrm{Hom}_{\mathbf{D}^-}(R_{tr}(X), C[p+q]).$$

Proof. This is well-known; see [Wei94, 5.7.9]. The spectral sequence is bounded, and hence converges, by 9.26. □

Proposition 9.28. *Let C be a bounded above cochain complex of étale sheaves of R-modules with transfer, where $cd_R(k) < \infty$. If the sheaves $a_{ét}H^n(C)$ are all strictly $\mathbb{A}^1$-homotopy invariant, then C is $\mathbb{A}^1$-local.*

Proof. Let C be a complex of étale sheaves with transfers. By 9.17, it suffices to prove that $\mathrm{cone}(f)$ is in this class when f is the projection $R_{tr}(X \times \mathbb{A}^1) \to R_{tr}(X)$. The map f induces a morphism between the spectral sequences of 9.27 for X and $X \times \mathbb{A}^1$. Because the sheaves $L = a_{ét}H^qC$ are strictly $\mathbb{A}^1$-homotopy invariant, they are $\mathbb{A}^1$-local by 9.23. Thus

$$\begin{aligned}\mathrm{Ext}^p(R_{tr}(X), L) &= \mathrm{Hom}_{\mathbf{D}^-}(R_{tr}(X)[-p], L)\\ &\cong \mathrm{Hom}_{\mathbf{D}^-}(R_{tr}(X \times \mathbb{A}^1)[-p], L) = \mathrm{Ext}^p(R_{tr}(X \times \mathbb{A}^1), L).\end{aligned}$$

Hence the morphism of spectral sequences is an isomorphism on all E_2 terms. By the Comparison Theorem [Wei94, 5.2.12], f induces an isomorphism from $\mathrm{Hom}_{\mathbf{D}^-}(R_{tr}(X)[n], C)$ to $\mathrm{Hom}_{\mathbf{D}^-}(R_{tr}(X \times \mathbb{A}^1)[n], C)$ for each n. Done. □

Lemma 9.29. *If K is a bounded above complex of étale sheaves of $\mathbb{Z}/n$-modules with transfer and $1/n \in k$, then $\mathrm{Tot}\, C_*(K)$ is $\mathbb{A}^1$-local.*

Proof. Set $C = \mathrm{Tot}\, C_*(K)$. By 2.18, each H^iC is an $\mathbb{A}^1$-homotopy invariant presheaf of $\mathbb{Z}/n$-modules with transfers. By the Rigidity Theorem 7.20, the sheaf $a_{ét}H^iC$ is locally constant. By 9.22, $a_{ét}H^iC$ is strictly $\mathbb{A}^1$-homotopy invariant. Finally, 9.28 lets us conclude that C is $\mathbb{A}^1$-local. □

Corollary 9.30. *If $1/n \in k$ then $\mathbb{Z}/n(q)$ is $\mathbb{A}^1$-local for all q.*

Proof. Take K to be $(\mathbb{Z}/n)_{tr}\mathbb{G}_m^{\wedge q}[-q]$; $\mathbb{Z}/n(q) = C_*K$ by definition 3.1. □

Definition 9.31. If $1/n \in k$, let $\mathcal{L}$ denote the full subcategory of $\mathbf{D}^-$ consisting of $\mathbb{A}^1$-local complexes of $\mathbb{Z}/n$-modules with transfer. If E and F are $\mathbb{A}^1$-local, we set $E \otimes_{\mathcal{L}} F = \operatorname{Tot} C_*(E \otimes^{tr}_{L,\acute{e}t} F)$. By 9.29, $E \otimes_{\mathcal{L}} F$ is $\mathbb{A}^1$-local, so $\otimes_{\mathcal{L}}$ is a bifunctor from $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$.

Recall from 6.9 that the category of locally constant étale sheaves of $\mathbb{Z}/n$-modules is equivalent to the category $\mathbf{Mod}(G, \mathbb{Z}/n)$ of discrete $\mathbb{Z}/n$-modules over the Galois group $G = Gal(k_{sep}/k)$. Let $\mathbf{D}^-(G, \mathbb{Z}/n)$ denote the (bounded above) derived category of such modules. There is a triangulated functor π^* from $\mathbf{D}^-(G, \mathbb{Z}/n)$ to $\mathbf{D}^- = \mathbf{D}^-(Sh_{\acute{e}t}(Cor_k, \mathbb{Z}/n))$.

Theorem 9.32. *If $1/n \in k$, $(\mathcal{L}, \otimes_{\mathcal{L}})$ is a tensor triangulated category and the functors*

$$\mathbf{D}^-(G, \mathbb{Z}/n) \xrightarrow{\pi^*} \mathcal{L} \longrightarrow \mathbf{D}^-[W_{\mathbb{A}}^{-1}] = \mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, \mathbb{Z}/n)$$

are equivalences of tensor triangulated categories.

Proof. Clearly, $\mathcal{L}$ is a thick subcategory of $\mathbf{D}^-$. By 9.20, the functor $\mathcal{L} \to \mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ is fully faithful. By 9.29, every object of $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ is isomorphic to an object of $\mathcal{L}$. Hence $\mathcal{L}$ is equivalent to $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ as a triangulated category.

By 9.5, $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k, \mathbb{Z}/n)$ is a *tensor* triangulated category. Using the first part of this proof, we conclude that $\mathcal{L}$ is a tensor triangulated category as well. Moreover, if E and F are $\mathbb{A}^1$-local, then $E \otimes_{\mathcal{L}} F$ is isomorphic to $E \otimes^{tr}_{L,\acute{e}t} F$ in $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ by 9.14, so the induced tensor operation on $\mathcal{L}$ is isomorphic to $\otimes_{\mathcal{L}}$.

Next we consider π^*. It is easy to see from 6.9 and 6.11 that π^* induces an equivalence between $\mathbf{D}^-(G, \mathbb{Z}/n)$ and the full subcategory of complexes of locally constant sheaves in $\mathbf{D}^-$. By exercise 8.20, π^* sends $\otimes^{\mathbb{L}}_{\mathbb{Z}/n}$ to $\otimes^{tr}_{L,\acute{e}t}$. It suffices to show that every $\mathbb{A}^1$-local complex F is isomorphic to such a complex. By 9.14, 9.29, and 9.18 $F \to C_*F$ is a quasi-isomorphism. By 2.18, each $a_{\acute{e}t}H^iF$ is $\mathbb{A}^1$-homotopy invariant. By 7.20 the sheaves $a_{\acute{e}t}H^iF$ are locally constant. Hence the canonical map $F \to \pi^*\pi_*F$ is a quasi-isomorphism of complexes of étale sheaves. But π_*F is a complex of modules in $\mathbf{Mod}(G, \mathbb{Z}/n)$. □

Lecture 10

Étale motivic cohomology and algebraic singular homology

There are two ways one might define an étale version of motivic cohomology. One way, which is natural from the viewpoint of these notes, is to use the morphisms in the triangulated category $\mathbf{DM}^-_{ét}$, namely to define the integral cohomology group indexed by (p,q) as $\mathrm{Hom}_{\mathbf{DM}^-_{ét}}(\mathbb{Z}_{tr}(X), \mathbb{Z}(q)[p])$, and similarly for cohomology with coefficients in an A. The second approach, due to Lichtenbaum, is to take the étale hypercohomology of the complex $\mathbb{Z}(q)$.

Definition 10.1. For any abelian group A, we define the **étale (or Lichtenbaum) motivic cohomology** of X as the hypercohomology of $A(q)$:

$$H^{p,q}_L(X,A) = \mathbb{H}^p_{ét}(X, A(q)|_{X_{ét}}).$$

If $q < 0$ then $H^{p,q}_L(X,A) = 0$, because $A(q) = 0$. If $q = 0$ then $H^{p,0}_L(X,A) \cong H^p_{ét}(X,A)$, because $A(0) = A$.

The two definitions agree in some cases of interest. We will see in 10.7 below that $H^{p,q}_L(X,\mathbb{Z}/n) \cong \mathrm{Hom}_{\mathbf{DM}^-_{ét}}(\mathbb{Z}_{tr}(X), \mathbb{Z}/n(q)[p])$ when $1/n \in k$. Even further on, in 14.21, we will see that $H^{p,q}_L(X,\mathbb{Q}) \cong \mathrm{Hom}_{\mathbf{DM}^-_{ét}}(\mathbb{Z}_{tr}(X), \mathbb{Q}(q)[p])$. However, the two definitions do not agree for ℓ-torsion coefficients, for $\ell = \mathrm{char}(k)$. Indeed, for $q = 0$ we have $\mathrm{Hom}_{\mathbf{DM}^-_{ét}}(\mathbb{Z}_{tr}(X), \mathbb{Z}/\ell[p]) = 0$ in characteristic ℓ by 9.15, yet the groups $H^{p,0}_L(X,\mathbb{Z}/\ell) \cong H^p_{ét}(X,\mathbb{Z}/\ell)$ can certainly be nonzero.

By proposition 6.4 we have $H^{p,1}_L(X,\mathbb{Z}/n) \cong H^p_{ét}(X,\mu_n)$ when $1/n \in k$. Here is the generalization to all q.

Theorem 10.2. *Let n be an integer prime to the characteristic of k then:*

$$H_L^{p,q}(X, \mathbb{Z}/n) = H^p_{\acute{e}t}(X, \mu_n^{\otimes q}) \qquad q \geq 0, p \in \mathbb{Z}$$

By 6.4 there is a quasi-isomorphism $\mu_n \to \mathbb{Z}/n(1)$ of complexes of étale sheaves. Because μ_n and the terms of $\mathbb{Z}/n(1)$ are flat as sheaves of $\mathbb{Z}/n$-modules, there is a morphism $\mu_n^{\otimes q} \to (\mathbb{Z}/n)(1)^{\otimes q}$ in the category of complexes of étale sheaves of $\mathbb{Z}/n$-modules. Combining with the multiplication of 3.10 gives a map

$$\mu_n^{\otimes q} \longrightarrow (\mathbb{Z}/n)(1)^{\otimes q} \longrightarrow (\mathbb{Z}/n)(q).$$

We may now reformulate theorem 10.2 as follows.

Theorem 10.3. *The map $\mu_n^{\otimes q} \to \mathbb{Z}/n(q)$ is a quasi-isomorphism of complexes of étale sheaves.*

Proof. The theorem is true for $q = 1$ by 6.4. By 9.24 and 9.29, both $\mu_n^{\otimes q}$ and $\mathbb{Z}/n(q)$ are $\mathbb{A}^1$-local. We will show that the map $\mu_n^{\otimes q} \to \mathbb{Z}/n(q)$ is an $\mathbb{A}^1$-weak equivalence in 10.6 below. By 9.18, it is also a quasi-isomorphism. $\square$

Let R be any commutative ring. Recall that $R(n) = R \otimes_{\mathbb{Z}} \mathbb{Z}(n)$. Clearly, the multiplication map $\mathbb{Z}(m) \otimes_{\mathbb{Z}} \mathbb{Z}(n) \to \mathbb{Z}(m+n)$ of 3.10 induces a map $R(m) \otimes_R R(n) \to R(m+n)$.

Proposition 10.4. *The multiplication map $R(m) \otimes R(n) \to R(m+n)$ factors through a map $\mu : R(m) \otimes^{tr} R(n) \to R(m+n)$.*

$$\begin{tikzcd}
R(m) \otimes_R R(n) \arrow[r, "\text{mult.}"] \arrow[d, "8.9"'] & R(m+n) \\
R(m) \otimes^{tr} R(n) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Proof. We first reinterpret the left vertical map in simplicial language. Recall that by definition 3.1, $R(n)[n] = C_*(R_{tr}(\mathbb{G}_m^{\wedge n}))$. Let us write $A^n_\bullet$ for the underlying simplicial presheaf, viz., $A^n_\bullet(U) = \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(U \times \Delta^\bullet)$, and write the associated unnormalized chain complex as A^n_*. By 8.9, we have a natural map of bisimplicial presheaves $A^m_\bullet \otimes_R A^n_\bullet \to A^m_\bullet \otimes^{tr} A^n_\bullet$, and a map of their diagonal chain complexes, $(A^m \otimes_R A^n)_* \to (A^m \otimes^{tr} A^n)_*$. As in

3.10, the Eilenberg-Zilber theorem yields quasi-isomorphisms ∇ fitting into a commutative diagram:

$$\begin{tikzcd}
R(m) \otimes_R R(n)[m+n] \arrow[r, "="] \arrow[d, "8.9"] & A^m_* \otimes_R A^n_* \arrow[r, "\nabla"] \arrow[d, "8.9"] & (A^m \otimes_R A^n)_* \arrow[d, "8.9"] \\
R(m) \otimes^{tr} R(n)[m+n] \arrow[r, "="] & A^m_* \otimes^{tr} A^n_* \arrow[r, "\nabla"] & (A^m \otimes^{tr} A^n)_*.
\end{tikzcd}$$

Comparing with 3.10, we see that if suffices to find a simplicial map for all X and Y,

$$\operatorname{diag}(C_\bullet R_{tr}(X) \otimes^{tr} C_\bullet R_{tr}(Y)) \longrightarrow C_\bullet R_{tr}(X \times Y) \tag{10.4.1}$$

compatible with the corresponding construction 3.9 for $\otimes_R$. The map μ will be the composite of ∇ and the map induced by 10.4.1.

Let F be any presheaf with transfers. Definitions 8.2 and 2.13 imply that $C_n(F) \cong \underline{Hom}(R_{tr}(\Delta^n), F)$ as presheaves and that $C_\bullet(F) \cong \underline{Hom}(R_{tr}(\Delta^\bullet_k), F)$ as simplicial presheaves. Using these identifications, we define the map 10.4.1 in degree n as the composition:

$$\begin{aligned}
&C_n(R_{tr}(X)) \otimes^{tr} C_n(R_{tr}(Y)) = \\
&\qquad \underline{Hom}(R_{tr}(\Delta^n), R_{tr}(X)) \otimes^{tr} \underline{Hom}(R_{tr}(\Delta^n), R_{tr}(Y)) \xrightarrow{8.5} \\
&\underline{Hom}(R_{tr}(\Delta^n \times \Delta^n), R_{tr}(X \times Y)) \xrightarrow{\text{diagonal}} \underline{Hom}(R_{tr}(\Delta^n), R_{tr}(X \times Y)) = \\
&\qquad\qquad\qquad\qquad C_n(R_{tr}(X \times Y)).
\end{aligned}$$

Since $\underline{Hom}(R_{tr}(\Delta^n \times \Delta^n), R_{tr}(X \times Y))(U) = R_{tr}(X \times Y)(U \times \Delta^n \times \Delta^n)$, the above composition is the right vertical composition in the following commutative diagram (see 8.5):

$$\begin{tikzcd}
R_{tr}(X)(U \times \Delta^n) \otimes_R R_{tr}(Y)(U \times \Delta^n) \arrow[r, "8.9"] \arrow[d, "\otimes"] & (C_n R_{tr}(X) \otimes^{tr} C_n R_{tr}(Y))(U) \arrow[d, "8.5"] \\
R_{tr}(X \times Y)(U \times U \times \Delta^n \times \Delta^n) \arrow[r, "\mathrm{diag}(U)"] \arrow[dr, "\mathrm{diag}(U \times \Delta^n)"'] & R_{tr}(X \times Y)(U \times \Delta^n \times \Delta^n) \arrow[d, "\mathrm{diag}(\Delta^n)"] \\
& R_{tr}(X \times Y)(U \times \Delta^n).
\end{tikzcd}$$

Since the left composite is the degree n part of construction 3.9, this shows that the triangle in 10.4 commutes. $\square$

Proposition 10.5. *The map $\mathbb{Z}/n(1)^{\otimes_L^{tr} q} \to \mathbb{Z}/n(q)$ is an $\mathbb{A}^1$-weak equivalence in $\mathbf{D}^-(Sh_{ét}(Cor_k, \mathbb{Z}/n))$.*

Proof. The assertion follows from the diagram in figure 10.1, remembering that by definition $\mathbb{Z}/n(q)$ is $C_*(\mathbb{Z}/n)_{tr}(\mathbb{G}_m^{\wedge n})[-q]$. $\square$

$$\begin{tikzcd}
\mathbb{Z}/n(1)^{\otimes_L^{tr} q} \arrow[rr] & & \mathbb{Z}/n(q) \\
(\mathbb{Z}/n)_{tr}(\mathbb{G}_m)[-1]^{\otimes_L^{tr} q} \arrow[u, "\simeq_{\mathbb{A}^1}", "9.4+9.14"'] \arrow[dr, "8.7", "\simeq"'] & & (\mathbb{Z}/n)_{tr}(\mathbb{G}_m^{\wedge q})[-q] \arrow[u, "9.14", "\simeq_{\mathbb{A}^1}"'] \\
& ((\mathbb{Z}/n)_{tr}(\mathbb{G}_m))^{\otimes^{tr} q}[-q] \arrow[ur, "8.10", "="'] &
\end{tikzcd}$$

Figure 10.1: The factorization in proposition 10.5

Proposition 10.6. *The map $\mu_n^{\otimes q} \to \mathbb{Z}/n(q)$ is an $\mathbb{A}^1$-weak equivalence in $\mathbf{D}^-(Sh_{ét}(Cor_k, \mathbb{Z}/n))$.*

Proof. Consider the following diagram, in which $\otimes^{tr}$ and $\otimes_L^{tr}$ are to be understood in $\mathbb{Z}/n$-modules.

$$\begin{tikzcd}
 & \mu_n^{\otimes_L^{tr} q} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathbb{Z}/n(1)^{\otimes_L^{tr} q} \arrow[d, "\mu_1"] \arrow[dr, "\mu_1 \circ \mu"] & \\
\mu_n^{\otimes q} \arrow[r, "\simeq"] & \mu_n^{\otimes^{tr} q} \arrow[r] & \mathbb{Z}/n(1)^{\otimes^{tr} q} \arrow[r, "\mu"'] & \mathbb{Z}/n(q)
\end{tikzcd}$$

We already know that the top map is a quasi-isomorphism by 6.4 and 8.16. Lemma 8.13 proves that the bottom left map $\mu_n^{\otimes q} \to \mu_n^{\otimes^{tr} q}$ is a quasi-isomorphism. Lemma 8.18 proves that the left vertical map is a quasi-isomorphism. Hence the assertion follows from proposition 10.5. $\square$

Recall that when $1/n \in k$ we have $\mathbf{DM}_{ét}^- = \mathbf{DM}_{ét}^{\mathrm{eff},-}(k, \mathbb{Z}/n)$.

Proposition 10.7. *If $1/n \in k$ then $H_L^{p,q}(X, \mathbb{Z}/n) \cong \operatorname{Hom}_{\mathbf{DM}_{ét}^-}(\mathbb{Z}_{tr}(X), \mathbb{Z}/n(q)[p])$.*

Proof. Since $A = \mathbb{Z}/n(q)$ is $\mathbb{A}^1$-local by 9.30, the right side is

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{DM}_{ét}^-}(\mathbb{Z}_{tr}(X), \mathbb{Z}/n(q)[p]) &= \operatorname{Hom}_{\mathbf{D}^-}(\mathbb{Z}_{tr}(X), \mathbb{Z}/n(q)[p]) \\
&= \operatorname{Ext}^p(\mathbb{Z}_{tr}(X), \mathbb{Z}/n(q)).
\end{aligned}$$

By 6.25, this Ext group is $H_{ét}^p(X, \mathbb{Z}/n(q))$, which is the left side. $\square$

As a bonus for all our hard work, we are able to give a nice interpretation of Suslin's algebraic singular homology. Recall that $R_{tr}(X) = \mathbb{Z}_{tr}(X) \otimes R$.

Definition 10.8. We define the algebraic singular homology of X by:

$$H_i^{sing}(X, R) = H_i(C_*(R_{tr}(X))(\operatorname{Spec} k).$$

By remark 7.4, $H_0^{sing}(X, \mathbb{Z})$ agrees with the group $H_0^{sing}(X/\operatorname{Spec} k)$ of lecture 7. As an exercise the reader should check that:

$$H^{p,q}(\operatorname{Spec} k, R) = H_{q-p}^{sing}(\mathbb{G}_m^{\wedge q}, R)$$

Notice that $R_{tr}(\mathbb{G}_m^{\wedge q})$ is well-defined even though $\mathbb{G}_m^{\wedge q}$ is not a scheme.

The following theorem was first proven in [SV96, 7.8] under the assumption of resolution of singularities on k. The proof we give here doesn't need resolution of singularities, so it extends the result to fields of positive characteristic.

Theorem 10.9. *Let k be a separably closed field and X a smooth scheme over k, and let l be a prime number different from* $\operatorname{char} k$. *Then there exist natural isomorphisms for all i:*

$$H_i^{sing}(X, \mathbb{Z}/l)^* \cong H^i_{ét}(X, \mathbb{Z}/l)$$

where the * denotes the dual vector space over $\mathbb{Z}/l$.

It is amusing to note that this implies that $H^i_{ét}(X, \mathbb{Z}/l)$ is finite, because it is a countable-dimensional dual module.

To prove 10.9, we need one more lemma. To clarify the role of the coefficient ring R, we will write $\mathbf{D}^-_R$ for $\mathbf{D}^-(Sh_{ét}(Cor_k, R))$, so that $\mathbf{D}^-_{\mathbb{Z}}$ is just the usual derived category of $Sh_{ét}(Cor_k)$.

Lemma 10.10. *Let k be a separably closed field and C a bounded above chain complex of étale sheaves of R-modules with transfer. Assume that the cohomology sheaves of C are locally constant and projective (as R-modules). Then for any $n \in \mathbb{Z}$ we have:*

$$\operatorname{Hom}_{\mathbf{D}^-_R}(C, R[n]) = \operatorname{Hom}_{R-\mathbf{mod}}(H^n(C)(\operatorname{Spec} k), R)$$

Proof. For simplicity, let us write Ext* for Ext in the category $Sh_{ét}(Cor_k, R)$. (There are enough injectives to define Ext by 6.19.)

If P is a summand of $\oplus_\alpha R$, then $\operatorname{Ext}^n(P, R)$ injects into

$$\operatorname{Ext}^n(\oplus_\alpha R, R) = \prod \operatorname{Ext}^n(R, R) = \prod \operatorname{Ext}^n(R_{tr}(\operatorname{Spec} k), R).$$

But $\operatorname{Ext}^n(R_{tr}(\operatorname{Spec} k), R) = H^n_{ét}(\operatorname{Spec} k, R)$ by 6.24 and this vanishes if $n \neq 0$ as k is separably closed. If $n = 0$, this calculation yields $\operatorname{Ext}^0(R, R) = R$ and $\operatorname{Ext}^0(P, R) = \operatorname{Hom}_{R-\mathbf{mod}}(P, R)$.

Now recall that $\operatorname{Ext}^n(F, R) = \operatorname{Hom}_{\mathbf{D}^-_R}(F, R[n])$ for every sheaf F; see [Wei94, 10.7.5]. More generally, if $R \to I^*$ is an injective resolution then the total Hom cochain complex $\mathbb{R}\operatorname{Hom}(C, R)$ of $\operatorname{Hom}^*(C, I[n])$ satisfies

$$H^n \mathbb{R}\operatorname{Hom}(C, R) \cong \operatorname{Hom}_{\mathbf{D}^-_R}(C, R[n]).$$

(See [Wei94, 10.7.4].) Since $\operatorname{Hom}^*(C, I[n])$ is a bounded double complex, it gives rise to a convergent spectral sequence which, as in [Wei94, 5.7.9], may be written

$$E_2^{pq} = \operatorname{Ext}^p(H^qC, R) \Longrightarrow H^{p+q}\mathbb{R}\operatorname{Hom}(C, R) = \operatorname{Hom}_{\mathbf{D}^-_R}(C, R[p+q]).$$

The assumption on H^qC makes the spectral sequence collapse to yield $\operatorname{Ext}^0(H^nC, R) \cong \operatorname{Hom}_{\mathbf{D}^-_R}(C, R[n])$, whence the result. ☐

Proof of 10.9. Taking $R = \mathbb{Z}/l$, this means that all R-modules are projective. Consider the diagram:

$$\begin{tikzcd} \operatorname{Hom}_{\mathbf{D}^-_R}(C_*(R_{tr}(X)), R[n]) \arrow[r, "\cong", "10.10"'] \arrow[d, "9.24"', "\cong"] & \operatorname{Hom}_{R-\mathbf{mod}}(H_n^{sing}(X, R), R) \\ \operatorname{Hom}_{\mathbf{D}^-_R}(R_{tr}(X), R[n]) \arrow[r, "\cong", "6.24"'] & H^n_{ét}(X, R). \end{tikzcd}$$

By 2.18, each $H^n = H^nC_*R_{tr}(X)$ is a homotopy invariant presheaf of $\mathbb{Z}/l$-modules with transfer. Hence the sheaves $a_{ét}H^n$ are locally constant by the Rigidity Theorem 7.20. Hence the top map is an isomorphism by 10.10. Since R is $\mathbb{A}^1$-local by 9.24, the left map is an isomorphism by 9.14. The bottom map is an isomorphism by 6.24. ☐

Corollary 10.11. *Let k be a separably closed field and X a smooth scheme over k, and let n be an integer relatively prime to* $\operatorname{char} k$*. Then there exist natural isomorphisms for all i:*

$$H_i^{sing}(X, \mathbb{Z}/n)^* \cong H^i_{ét}(X, \mathbb{Z}/n)$$

where the * denotes the Pontrjagin dual $\mathbb{Z}/n$-module.

Proof. Using the sequences $0 \to \mathbb{Z}/l \to \mathbb{Z}/lm \to \mathbb{Z}/m \to 0$, the 5-lemma shows that we may assume that n is prime. ☐

Lecture 11

Standard triples

For all of this lecture, F will be a homotopy invariant presheaf with transfers.

Our goal in this lecture is to prove the following result, which is one of the main properties of homotopy invariant presheaves with transfers. It (or rather its corollary 11.2) will be used in subsequent lectures to promote results from the Nisnevich topology to the Zariski topology. It depends primarily upon the relative Picard group introduced in lecture 7.

Recall that a subgroup A of an abelian group B is called *pure* if $nA = nB \cap A$ for every integer n. A homomorphism $f : A \to B$ of abelian groups is called *pure injective* if it is injective and $f(A)$ is a pure subgroup of B.

Proposition 11.1. *For any smooth semilocal S over k, any Zariski dense open subset $V \subset S$, and any homotopy invariant presheaf with transfers F, the map $F(S) \to F(V)$ is pure injective.*

Proposition 11.1 is a consequence of a more precise result, proposition 11.3, whose proof will take up most of this lecture.

Proof. The semilocal scheme S is the intersection of a family X_α of smooth varieties of finite type over k and V is the intersection of dense open subschemes $V_\alpha \subset X_\alpha$. Hence $F(S) \to F(V)$ is the filtered colimit of the maps $F(X_\alpha) \to F(V_\alpha)$. Since all these maps are injections by 11.3, their colimit is an injection. If $a \in F(X_\alpha)$ equals $nb \in F(V_\alpha)$ for some $b \in F(V_\alpha)$, then the image of a in $F(U_\alpha)$, and hence in $F(S)$, is n-divisible. $\square$

Passing to the direct limit over all such V, we see that $F(S)$ injects (as a pure subgroup) into the direct sum of the $F(\operatorname{Spec} E_i)$, as E_i runs over the generic points of any semilocal S. In particular, we have:

Corollary 11.2. *Let F be a homotopy invariant presheaf with transfers. If $F(\operatorname{Spec} E) = 0$ for every field E over k, then $F_{Zar} = 0$.*

Theorem 11.3. *Let X be smooth of finite type over a field k and let V be a dense open subset. Then for every finite set of points $x_1, \ldots, x_n \in X$ there exists an open neighborhood U of these points such that $F(X) \to F(U)$ factors through $F(X) \to F(V)$. That is, there is a map $F(V) \to F(U)$ such that the following diagram commutes.*

$$\begin{tikzcd} F(X) \arrow[d] \arrow[dr] & \\ F(V) \arrow[r, "\exists"] & F(U) \end{tikzcd}$$

Example 11.4. If $V \subsetneqq X$ is a dense open subset, then $F = \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(V)$ is a presheaf with transfers, but $F(X) \to F(V)$ is not injective. (1_X is nonzero in $F(X)$ but vanishes in $F(V)$.) This shows that homotopy invariance is necessary in 11.3.

To prepare for the proof of proposition 11.3, we need a technical digression.

Definition 11.5. A **standard triple** is a triple $(\bar{X} \xrightarrow{\bar{p}} S, X_\infty, Z)$ where $\bar{p}$ is a proper morphism of relative dimension 1 and Z and X_∞ are closed subschemes of $\bar{X}$. The following conditions must be satisfied:

1. S is smooth and $\bar{X}$ is normal,
2. $\bar{X} - X_\infty$ is quasi-affine and smooth over S,
3. $Z \cap X_\infty = \emptyset$,
4. $X_\infty \cup Z$ lies in an affine open neighborhood in $\bar{X}$.

Given a standard triple as above, we usually write X for $\bar{X} - X_\infty$. Note that $\bar{X}$ is a good compactification of both X and $X - Z$ (see 7.8) by parts 2 and 4.

Conversely, if $\bar{X}$ is a good compactification of a smooth quasi-affine curve $X \to S$ (see 7.8), then $(\bar{X}, \bar{X} - X, \emptyset)$ is a standard triple.

We will see in 11.17 below that any pair of smooth quasi-projective varieties $Z \subset X$ is locally part of a standard triple, at least when k is infinite.

Remark 11.6. (Gabber) Parts 4 and 2 imply that S is affine, and that Z and X_∞ are finite over S. Indeed, X_∞ is finite and surjective over S by part 2, and affine by part 4, so Chevalley's theorem ([Har77, III Ex.4.2]) implies that S is affine.

We will make use of the following observation. Recall from 7.10 that $\mathrm{Pic}(\bar{X}, X_\infty)$ is the group of isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle on $\bar{X}$ and s is a trivialization on X_∞.

Given a standard triple $(\bar{X}, X_\infty, Z)$, any section $x : S \to X$ of p defines an element $[x]$ of $\mathrm{Pic}(\bar{X}, X_\infty)$. Indeed, there is a homomorphism $C_0(X/S) \to \mathrm{Pic}(\bar{X}, X_\infty)$.

Remark 11.7. Let F be a homotopy invariant presheaf with transfers. Given a standard triple $(\bar{X}, X_\infty, Z)$, by 7.5 there is a pairing:

$$(\ , \ ) : \mathrm{Pic}(\bar{X}, X_\infty) \otimes F(X) \to F(S).$$

Let $x : S \to X$ be a section of p. If $[x]$ is the class of x in $\mathrm{Pic}(\bar{X}, X_\infty)$, then $([x], f) = F(x)(f)$ for all $f \in F(X)$.

Lemma 11.8. *Let $(\bar{X}, X_\infty, Z)$ be a standard triple over S and $X = \bar{X} - X_\infty$. Then there is a commutative diagram for every homotopy invariant presheaf with transfers F.*

$$\begin{tikzcd}
\mathrm{Pic}(\bar{X}, X_\infty \coprod Z) \otimes F(X) \arrow[r] \arrow[d] & \mathrm{Pic}(\bar{X}, X_\infty \coprod Z) \otimes F(X - Z) \arrow[d] \\
\mathrm{Pic}(\bar{X}, X_\infty) \otimes F(X) \arrow[r] & F(S)
\end{tikzcd}$$

Proof. Be definition, $\bar{X}$ is a good compactification of both X and $X - Z$. Thus the pairings exist by 7.5 (or 7.16) and are induced by the transfers pairing $Cor_k(S, X) \otimes F(X) \to F(S)$. Commutativity of the diagram is a restatement of the fact that any presheaf with transfers is a functor on Cor_k. $\square$

Corollary 11.9. *If $x : S \to X$ is a section and $[x] \in \mathrm{Pic}(\bar{X}, X_\infty)$ lifts to $\lambda \in \mathrm{Pic}(\bar{X}, X_\infty \coprod Z)$, there is a commutative diagram:*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "{[x]}"'] & F(X - Z) \arrow[dl, "\lambda"] \\ F(S). & \end{tikzcd}$$

Moreover, if $\lambda' \in C_0(X - Z/S) \subset Cor(S, X - Z)$ is any representative of λ (see 7.16 and 1A.10), the composition of λ' with the inclusion $X - Z \subset X$ is $\mathbb{A}^1$-homotopic to x in $Cor(S, X)$.

Exercise 11.10. Use example 7.14 with $F = \mathcal{O}^*$ to show that there can be more than one lift $\lambda : F(X - Z) \to F(S)$.

More generally, observe that any unit s of $\mathcal{O}(Z)$ gives a trivialization of $\mathcal{O}(\bar{X})$ on Z; combining this with the trivialization 1 on X_∞ gives an element $\sigma(s) = (\mathcal{O}, 1 \coprod s)$ of $\mathrm{Pic}(\bar{X}, X_\infty \coprod Z)$. Show that $\lambda + \sigma(s)$ is also a lift of $[x]$ to $\mathrm{Pic}(\bar{X}, X_\infty \coprod Z)$, and that every other lift has this form for some $s \in \mathcal{O}^*(Z)$.

Definition 11.11. A standard triple is **split** over an open subset $U \subset X$ if $\mathcal{L}_\Delta|_{U \times_S Z}$ is trivial, where $\mathcal{L}_\Delta$ is the line bundle on $U \times_S \bar{X}$ corresponding to the graph of the diagonal map.

Example 11.12. For any affine S, the standard triple $(S \times \mathbb{P}^1, S \times \infty, S \times 0)$ is split over any U in $X = S \times \mathbb{A}^1$. Indeed, the line bundle $\mathcal{L}_\Delta$ is trivial on all of $X \times X$.

Exercise 11.13. Let $\bar{X}$ be a smooth projective curve over k, with affine open $X = \mathrm{Spec}(A)$ and set $X_\infty = \bar{X} - X$. Then $(\bar{X}, X_\infty, Z)$ is a standard triple for every finite Z in X. Let $P_1, \ldots$ be the prime ideals of A defining the points of Z, and suppose for simplicity that $A/P_i \cong k$ for all i. Show that the standard triple splits over $D(f)$ if and only if each P_i becomes a principal ideal in the ring $A[1/f]$.

In particular, if $\bar{X} = \mathbb{P}^1$, the triple splits over all X because in this case A is a principal ideal domain.

Lemma 11.14. *Any finite set of points in X has an open neighborhood U such that the triple is split over U.*

Proof. The map $f : X \times_S Z \to X$ is finite, as Z is finite over S. Given points $x_i \in X$, each $f^{-1}(x_i)$ is finite. Now the line bundle $\mathcal{L}_\Delta$ is trivial in some neighborhood V of $\cup_i f^{-1}(x_i)$, because every line bundle on a semilocal scheme is trivial. But every such V contains an open of the form $U \times_S Z$, and the triple is split over such a U. $\square$

Proposition 11.15. *Consider a standard triple split over an affine U. Then there is an $\mathbb{A}^1$-equivalence class of finite correspondences $\lambda : U \to (X - Z)$ such that the composite of λ with $(X - Z) \subset X$ is $\mathbb{A}^1$-homotopic to the inclusion $U \subset X$.*

In particular, $F(X) \to F(U)$ factors through $\lambda : F(X - Z) \to F(U)$:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X-Z) \arrow[dl, "\exists\lambda"] \\ F(U). & \end{tikzcd}$$

Proof. Pulling back yields a standard triple $(U \times_S \bar{X}, U \times_S X_\infty, U \times_S Z)$ over the affine U. The diagonal $\Delta : U \to U \times_S X$ is a section and its class in $\mathrm{Pic}(U \times_S \bar{X}, U \times_S X_\infty)$ is represented by the line bundle $\mathcal{L}_\Delta$. If the triple is split over an affine U, then $\mathcal{L}_\Delta$ has a trivialization on $U \times_S Z$ as well, so $[\Delta]$ lifts to a class λ in $\mathrm{Pic}(U \times_S \bar{X}, U \times_S (X_\infty \coprod Z))$. By 7.2 and 7.16, λ is an $\mathbb{A}^1$-equivalence class of maps in $Cor(U, X - Z)$. By 11.9 we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\lambda"] \arrow[dr, "{[\Delta]}"'] & U \times_S (X - Z) \arrow[r, "{[pr]}"] \arrow[d] & X - Z \arrow[d] \\ & U \times_S X \arrow[r, "{[pr]}"] & X \end{tikzcd}$$

and it suffices to observe that $pr \circ \Delta : U \to U \times_S X \to X$ is the inclusion. $\square$

A different splitting (trivialization on $U \times_S Z$) may yield a different lifting λ'. By exercise 11.10, $\lambda' = \lambda + \sigma(s)$ for some unit s of $\mathcal{O}(U \times_S Z)$.

Exercise 11.16. Suppose that λ is represented by an element D of $Cor(U, X - Z) = C_0(U \times (X - Z)/U)$, as in exercise 7.15. Show that the element $D - [\Delta(U)]$ of $Cor(U, X)$ is represented by a principal divisor (f) on $U \times \bar{X}$, with f equal to 1 on $U \times X_\infty$.

Theorem 11.17. *Let W be a connected quasi-projective smooth scheme over an infinite field k, Y a proper closed subset of W and $y_1, \dots, y_n \in Y$. Then there is an affine open neighborhood X of these points in W and a standard triple $(\bar{X} \to S, X_\infty, Z)$ such that $(X, X \cap Y) \cong (\bar{X} - X_\infty, Z)$.*

Proof. (Mark Walker) We may assume that W is affine, a closed $(d+1)$-dimensional subscheme of $\mathbb{A}^n$. Embed $\mathbb{A}^n$ in $\mathbb{A}^N$ by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_ix_j, \dots, x_n^2).$$

Given a closed point $x \in W$, Bertini's Theorem (see [SGA4, XI.2.1]) implies that the general linear projection $p : \mathbb{A}^N \to \mathbb{A}^d$ is smooth near each point of W lying on $p^{-1}(p(x))$. It is also finite when restricted to Y, because Y has dimension $\le d$.

Let $\bar{W}$ denote the closure of W in $\mathbb{P}^N$, $H = \mathbb{P}^N - \mathbb{A}^N$, and $W_\infty = \bar{W} \cap H$. The general projection defines a rational map $p : \bar{W} \dashrightarrow \mathbb{P}^d$ whose center C is finite, because C lies in the intersection of W_∞ with a codimension d linear subspace of H. Let $\bar{X}_1$ be the closure of the graph of $p : (\bar{W} - C) \to \mathbb{P}^d$ in $\bar{W} \times \mathbb{P}^d$. Then W is naturally an open subscheme of $\bar{X}_1$ and $\bar{X}_1 - W$ has finite fibers over $\mathbb{A}^d$.

The singular points Σ of the projection $\bar{X}_1 \to \mathbb{P}^d$ are closed, and finite over each $p(y_i)$ because p is smooth near $W \cap p^{-1}(p(y_i))$. Therefore there is an affine open neighborhood S in $\mathbb{A}^d$ of $\{p(y_i)\}$ over which Σ is finite and disjoint from Y. Define X to be $p^{-1}(S) \cap W - \Sigma$; by construction $p : X \to S$ is smooth. Define $\bar{X} \subset \bar{X}_1$ to be the inverse image of S, and $X_\infty = \bar{X} - X$. Then $X \cap Y \to S$ and $X_\infty \to S$ are both finite.

It remains to show that $X_\infty \coprod (X \cap Y)$ lies in an affine open neighborhood of $\bar{X}$. As $\bar{X}$ is projective over S, there is a global section of some very ample line bundle $\mathcal{L}$ whose divisor D misses all of the finitely many points of X_∞ and $X \cap Y$ over any $p(y_i)$. Because $\mathcal{L}$ is very ample and S is affine, $\bar{X} - D$ is affine. Replacing S by a smaller affine neighborhood of the $p(y_i)$, we can assume that D misses X_∞ and $X \cap Y$, i.e., that X_∞ and $X \cap Y$ lie in $\bar{X} - D$, as desired. $\square$

Porism 11.18. If k is finite, the proof shows that there is a finite extension k' and an affine open X' of the points in $W \times_k \operatorname{Spec} k'$ so that $(X', X' \cap Y')$ comes from a standard triple over k', where $Y' = Y \times_k \operatorname{Spec} k'$. In fact, for each prime l we can assume that $[k' : k]$ is a power of l.

Finally, we will use 11.15, 11.14 and 11.17 to prove 11.3.

Proof of 11.3. We first assume that k is infinite. Since we may replace V by $V - \{x_1, \ldots, x_n\}$, we may assume that the closed points $x_1, \ldots, x_n$ of X lie in $Z = X - V$. We can use 11.17 to shrink X about these points to assume that there exists a standard triple with $X = \bar{X} - X_\infty$. By 11.14 the triple splits over an open neighborhood U of the points. As X is quasi-projective, we may shrink U to make it affine. By 11.15 we get the map $F(X - Z) \to F(U)$ factoring $F(X) \to F(U)$.

If k is finite, we proceed as follows. We see by porism 11.18 that there is an open X' of $X \times_k \operatorname{Spec}(k')$ fitting into a standard triple over k'. The argument above shows that there is an open neighborhood U of $x_1, \ldots, x_n$ (depending on k') such that if $U' = U \times_k \operatorname{Spec}(k')$ and $V' = V \times_k \operatorname{Spec}(k')$, then $F(X') \to F(U')$ factors through a map $\Phi' : F(V') \to F(U')$. Let $\Phi(k') : F(V) \to F(U)$ be the composite of Φ' and the transfer $F(U') \to F(U)$. By 1.11, $[k' : k]$ times $F(X) \to F(U)$ factors through $\Phi(k')$. By 11.18, we can choose two such extensions k',k'' with $[k' : k]$ and $[k'' : k]$ relatively prime. Shrinking U, we may assume that $F(U)$ is the target of both $\Phi(k')$ and $\Phi(k'')$. But then $F(X) \to F(U)$ factors through a linear combination of $\Phi(k')$ and $\Phi(k'')$. $\square$

Lecture 12

Nisnevich sheaves

We have already mentioned the Nisnevich topology several times in previous lectures, as an alternative to the étale and Zariski topologies. In this lecture we develop some of its more elementary properties.

We begin by recalling the definition of the Nisnevich topology (see [Nis89]). A family of étale morphisms $\{p_i : U_i \to X\}$ is said to be a *Nisnevich covering* of X if it has the Nisnevich lifting property:

- for all $x \in X$, there is an i and a $u \in U_i$ so that $p_i(u) = x$ and the induced map $k(x) \to k(u)$ is an isomorphism.

It is easy to check that this notion of cover satisfies the axioms for a Grothendieck topology (in the sense of [Mil80, I.1.1], or pre-topology in the sense of [SGA4]). The Nisnevich topology is the class of all Nisnevich coverings.

Example 12.1. Here is an example to illustrate the arithmetic nature of a Nisnevich cover. When $\operatorname{char} k \neq 2$, the two morphisms $U_0 = \mathbb{A}^1 - \{a\} \xhookrightarrow{j} \mathbb{A}^1$ and $U_1 = \mathbb{A}^1 - \{0\} \xrightarrow{z \mapsto z^2} \mathbb{A}^1$ form a Nisnevich covering of $\mathbb{A}^1$ if and only if $a \in (k^*)^2$. They form an étale covering of $\mathbb{A}^1$ for any nonzero $a \in k$.

Example 12.2. Let k be a field. The small Nisnevich site on $\operatorname{Spec} k$ consists of the étale U over $\operatorname{Spec} k$, together with their Nisnevich coverings. Every étale U over $\operatorname{Spec} k$ is a finite disjoint union $\coprod \operatorname{Spec} l_i$ with the l_i finite and separable over k; to be a Nisnevich cover, one of the l_i must equal k. Thus a Nisnevich sheaf F on $\operatorname{Spec} k$ merely consists of a family of sets $F(l)$, natural in the finite separable extension fields l of k. In fact, each such l determines a "point" of $(\operatorname{Spec} k)_{Nis}$ in the sense of [SGA4, IV 6.1].

From this description it follows that $\operatorname{Spec} k$ has Nisnevich cohomological dimension zero. This implies that the Nisnevich cohomological dimension of any Noetherian scheme X is at most $\dim X$; see [KS86].

Lemma 12.3. *If $\{U_i \to X\}$ is a Nisnevich covering then there is a nonempty open $V \subset X$ and an index i such that $U_i|_V \to V$ has a section.*

Proof. For each generic point x of X, there is a generic point $u \in U_i$ so that $k(x) \cong k(u)$. Hence $U_i \to X$ induces a rational isomorphism between the corresponding components of U_i and X, i.e., $U_i \to X$ has a section over an open subscheme V of X containing x. $\square$

Example 12.4. A **Hensel local ring** $(R, \mathfrak{m})$ is a local ring such that any finite R-algebra S is a product of local rings. It is well-known (see [Mil80, I.4.2]) that if S is finite and étale over R, and if $R/\mathfrak{m} \cong S/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ of S, then $R \to S$ splits; one of the factors of S is isomorphic to R. If $\{U_i \to \operatorname{Spec} R\}$ is a Nisnevich covering then some U_i is finite étale, so $U_i \to \operatorname{Spec} R$ splits. Thus every Nisnevich covering of $\operatorname{Spec} R$ has the trivial covering as a refinement. Consequently, the Hensel local schemes $\operatorname{Spec} R$ determine "points" for the Nisnevich topology.

As with any Grothendieck topology, the category $Sh_{Nis}(Sm/k)$ of Nisnevich sheaves of abelian groups is abelian, and sheafification $F \mapsto F_{Nis}$ is an exact functor. We know that exactness in $Sh_{Nis}(Sm/k)$ may be tested at the Hensel local rings $\mathcal{O}^h_{X,x}$ of all smooth X at all points x (see [Nis89, 1.17]). That is, for every presheaf F:

- $F_{Nis} = 0$ if and only if $F(\operatorname{Spec} \mathcal{O}^h_{X,x}) = 0$ for all (X, x);
- $F_{Nis}(\operatorname{Spec} \mathcal{O}^h_{X,x}) = F(\operatorname{Spec} \mathcal{O}^h_{X,x})$.

By abuse of notation, we shall write $F(\mathcal{O}^h_{X,x})$ for $F(\operatorname{Spec} \mathcal{O}^h_{X,x})$, and refer to it as the *stalk* of F_{Nis} at x.

Definition 12.5. A commutative square $Q = Q(X, Y, A)$ of the form

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[d, "f"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

is called **upper distinguished** if $B = A \times_X Y$, f is étale, $i : A \to X$ is an open embedding and $(Y - B) \to (X - A)$ is an isomorphism. Clearly, any upper distinguished square determines a Nisnevich covering of X: $\{Y \to X, A \to X\}$.

Exercise 12.6. If $\dim X \leq 1$ show that any Nisnevich cover of X admits a refinement $\{U, V\}$ such that $Q(X, U, V)$ is upper distinguished. Show that this fails if $\dim X \geq 2$.

By definition, $F(Q)$ is a pullback square if and only if $F(X)$ is the pullback $F(Y) \times_{F(B)} F(A)$, i.e., the kernel of $f - i : F(Y) \times F(A) \to F(B)$.

Lemma 12.7. *A presheaf F is a Nisnevich sheaf if and only if $F(Q)$ is a pull-back square for every upper distinguished square Q.*

Proof. For the "if" part, suppose that each $F(Q)$ is a pullback square. To prove that F is a Nisnevich sheaf, fix a Nisnevich covering $\{U_i \to X\}$. Let us say that an open subset $V \subset X$ is *good* (for the covering) if

$$F(V) \longrightarrow \prod F(U_i \times_X V) \rightrightarrows \prod F(U_i \times_X U_j \times_X V)$$

is an equalizer diagram. We need to show that X itself is *good*.

By Noetherian induction, we may assume that there is a largest *good* $V \subset X$. Suppose that $V \neq X$ and let $Z = X - V$. By lemma 12.3, there is a nonempty open $W \subset Z$ and an index i such that $U_i|_W \to W$ splits. Let $X' \subset X$ be the complement of the closed set $Z - W$. Then V and $U_i' = U_i|_{X'}$ form an upper distinguished square Q over X'. Pulling back along each $U_j' = U_j|_{X'}$ also yields an upper distinguished square. Thus we have pullback squares

$$\begin{array}{ccc} F(X') & \longrightarrow & F(U_i') \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U_i'|_V) \end{array} \qquad \begin{array}{ccc} F(U_j') & \longrightarrow & F(U_i' \times_X U_j') \\ \downarrow & & \downarrow \\ F(U_j'|_V) & \longrightarrow & F(U_i' \times_X U_j'|_V). \end{array}$$

A diagram chase shows that X' is also *good*, contradicting the assumption that $V \neq X$. Hence X is *good* for each cover, i.e., F is a Nisnevich sheaf.

For "only if", we assume that F is a Nisnevich sheaf and Q is upper distinguished and need to prove that the map $F(X) \to F(Y) \times_{F(B)} F(A)$

is an isomorphism. We already know the map is monic because $\{A, Y\}$ is a Nisnevich cover of X. For the surjectivity, note that the sheaf axiom for this covering yields the equalizer sequence

$$F(X) \to F(Y) \times F(A) \rightrightarrows F(B) \times F(A) \times F(Y \times_X Y).$$

Since $\{\Delta(Y), B \times_A B\}$ is a cover of $Y \times_X Y$, we have an injection $F(Y \times_X Y) \to F(Y) \times F(B \times_A B)$. Now $(a, y) \in F(A) \times F(Y)$ lies in $F(A) \times_{F(B)} F(Y)$ if the two restrictions to $F(B)$ are the same. The two maps to $F(A)$ and $F(Y)$ are the same, so it suffices to consider the maps from $F(Y)$ to $F(B \times_A B)$. These both factor through $F(B)$, so the images of y are the same as the images of a. But by construction the two maps $F(A) \rightrightarrows F(B \times_A B)$ are the same. $\square$

Porism 12.8. Suppose more generally that F is a sheaf for some Grothendieck topology, and that $Q = Q(X, Y, A)$ is a pullback square whose horizontal maps are monomorphisms. If $\{A, Y\}$ is a cover of X and $\{B \times_A B, Y\}$ is a cover of $Y \times_X Y$, the proof of 12.7 shows that $F(Q)$ is a pullback square.

Exercise 12.9. Write $\mathcal{O}^*/\mathcal{O}^{*l}$ for the presheaf $U \mapsto \mathcal{O}^*(U)/\mathcal{O}^{*l}(U)$, and $\mathcal{O}^*/l$ for the Zariski sheaf associated to $\mathcal{O}^*/\mathcal{O}^{*l}$. Show that there is an exact sequence

$$0 \to \mathcal{O}^*(U)/\mathcal{O}^{*l}(U) \to \mathcal{O}^*/l(U) \to \mathrm{Pic}(U) \xrightarrow{\ l\ } \mathrm{Pic}(U)$$

for all smooth U. Then show that $\mathcal{O}^*/l$ is a Nisnevich sheaf on Sm/k. If $1/l \in k$, this is an example of a Nisnevich sheaf which is not an étale sheaf. In fact, $(\mathcal{O}^*/l)_{ét} = 0$.

Exercise 12.10. If F is a Nisnevich sheaf, consider the presheaf $E^0(F)$ defined by:

$$E^0(F)(X) = \prod_{\substack{\text{closed} \\ x \in X}} F(\mathcal{O}^h_{X,x}).$$

Show that $E^0(F)$ is a Nisnevich sheaf, and that the canonical map $F \to E^0(F)$ is an injection. Using 12.2, show that $E^0(F)$ is a flasque sheaf, i.e., that it has no higher cohomology (see [SGA4, V.4.1]). Iteration of this construction yields the canonical flasque resolution $0 \to F \to E^0(F) \to \cdots$ of a Nisnevich sheaf, which may be used to compute the cohomology groups $H^*_{Nis}(X, F)$.

Definition 12.11. Consider the presheaf sending U to $\mathbb{Z}[\mathrm{Hom}_{Sm/k}(U,X)]$. We write $\mathbb{Z}(X)$ for $\mathbb{Z}[\mathrm{Hom}(-,X)]_{Nis}$, its sheafification with respect to the Nisnevich topology. It is easily checked that $\mathbb{Z}[\mathrm{Hom}(-,X)]_{Zar}(U) = \mathbb{Z}[\mathrm{Hom}(-,X)](U)$ for every connected open U. We do not know if this is true for $\mathbb{Z}(X)$.

By the Yoneda lemma, $\mathrm{Hom}(\mathbb{Z}(X),G) = G(X)$ for every sheaf G. Since $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf by 6.2, we see that $\mathbb{Z}(X)$ is a subsheaf of $\mathbb{Z}_{tr}(X)$.

Let $\mathbf{D}^-_{Nis}$ denote the derived category of cohomologically bounded above complexes in $Sh_{Nis}(Sm/k)$. If F and G are Nisnevich sheaves, it is well known that $\mathrm{Ext}^n_{Nis}(F,G) = \mathrm{Hom}_{\mathbf{D}^-_{Nis}}(F,G[n])$ (see [Wei94, 10.7.5]).

Lemma 12.12. *Let G be any Nisnevich sheaf. Then for all X:*

$$\mathrm{Ext}^n_{Nis}(\mathbb{Z}(X),G) = H^n_{Nis}(X,G).$$

Proof. If $G \to I^*$ is a resolution by injective Nisnevich sheaves, then the nth cohomology of G is H^n of $I^*(X)$. But by [Wei94, 10.7.4] we know that the left side is H^n of $\mathrm{Hom}_{Sh_{Nis}(Sm/k)}(\mathbb{Z}(X),I^*) = I^*(X)$. ☐

Lemma 12.13. *The smallest class in $\mathbf{D}^-_{Nis}$ which contains all the $\mathbb{Z}(X)$ and is closed under quasi-isomorphisms, direct sums, shifts, and cones is all of $\mathbf{D}^-_{Nis}$.*

Proof. The proof of 9.3 goes through using $\mathbb{Z}(X)$ in place of $R_{tr}(X)$. ☐

For the rest of this lecture, we shall write $\otimes$ for the presheaf tensor product, $(F \otimes G)(U) = F(U) \otimes_{\mathbb{Z}} G(U)$, and $\otimes_{Nis}$ for the tensor product of Nisnevich sheaves, i.e., the sheafification of $\otimes$. Note that if a sheaf F is flat as a presheaf then F is also flat as a sheaf. This is true for example of the sheaves $\mathbb{Z}(X)$.

Lemma 12.14. *$\mathbb{Z}(X \times Y) = \mathbb{Z}(X) \otimes_{Nis} \mathbb{Z}(Y)$.*

Proof. Since $\mathrm{Hom}(U, X \times Y) = \mathrm{Hom}(U,X) \times \mathrm{Hom}(U,Y)$, we see that $\mathbb{Z}[\mathrm{Hom}(U,X \times Y)] = \mathbb{Z}[\mathrm{Hom}(U,X)] \otimes \mathbb{Z}[\mathrm{Hom}(U,Y)]$. Thus $\mathbb{Z}[\mathrm{Hom}(-,X \times Y)] \cong \mathbb{Z}[\mathrm{Hom}(-,X)] \otimes \mathbb{Z}[\mathrm{Hom}(-,Y)]$ as presheaves. Now sheafify. ☐

Lemma 12.15. *Let G be a Nisnevich sheaf on Sm/k such that $H^n_{Nis}(-,G)$ is homotopy invariant for all n. Then for all n and all bounded above C:*

$$\mathrm{Hom}_{\mathbf{D}^-_{Nis}}(C,G[n]) \cong \mathrm{Hom}_{\mathbf{D}^-_{Nis}}(C \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1), G[n]).$$

Proof. By 12.12, our assumption yields $\mathrm{Ext}^n(\mathbb{Z}(X), G) \cong \mathrm{Ext}^n(\mathbb{Z}(X \times \mathbb{A}^1), G)$ for all X. Since $\mathbb{Z}(X \times \mathbb{A}^1) = \mathbb{Z}(X) \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1)$ by 12.14, the conclusion holds for $C = \mathbb{Z}(X)$. If C and C' are quasi-isomorphic, then so are $C \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1)$ and $C \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1)$, because $\mathbb{Z}(\mathbb{A}^1)$ is a flat sheaf. But the class of all complexes C for which the conclusion holds is closed under direct sums and cones, so by 12.13 the conclusion holds for all C. $\square$

We borrow yet another topological definition: deformation retract. For each F, note that the presheaf $F \otimes \mathbb{Z}[\mathrm{Hom}(-, \operatorname{Spec} k)]$ is just F.

Definition 12.16. An injection of presheaves $i : F \to G$ is called a **(strong) deformation retract** if there is a map $r : G \to F$ such that $r \circ i = id_F$ and a homotopy $h : G \otimes \mathbb{Z}[\mathrm{Hom}(-, \mathbb{A}^1)] \to G$ so that the restriction $h|_F$ is the projection $F \otimes \mathbb{Z}[\mathrm{Hom}(-, \mathbb{A}^1)] \longrightarrow F$, $h(G \otimes 0) = i \circ r$ and $h(G \otimes 1) = id$.

If F and G are sheaves, the condition in the definition is equivalent to the condition that there is a sheaf map $h : G \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1) \to G$ so that the restriction $h|_F$ is the projection $F \otimes_{Nis} \mathbb{Z}(\mathbb{A}^1) \to F$, $h(G \otimes 0) = i \circ r$ and $h(G \otimes 1) = id$.

For example, the zero-section $\operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1$ induces a deformation retract $\mathbb{Z} \to \mathbb{Z}(\mathbb{A}^1)$; the homotopy map h is induced by the multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ using 12.14. If I^1 is the quotient presheaf $\mathbb{Z}(\mathbb{A}^1)/\mathbb{Z}$, so that $\mathbb{Z}(\mathbb{A}^1) \cong \mathbb{Z} \oplus I^1$, then $0 \subset I^1$ is also a deformation retract.

Lemma 12.17. *If $F \to G$ is a deformation retract, then the quotient presheaf G/F is a direct summand of $G/F \otimes I^1$.*

Proof. The inclusion $0 \subset G/F$ is a deformation retract, whose homotopy is induced from h. Therefore we may assume that $F = 0$.

Let K denote the kernel of h. Since the evaluation "$t = 1$" $: G = G \otimes \mathbb{Z} \to G \otimes \mathbb{Z}(\mathbb{A}^1)$ is a section of both h and the projection $G \otimes \mathbb{Z}(\mathbb{A}^1) \to G$, we see that K is isomorphic to $G \otimes I^1$. But "$t = 0$": $G \to G \otimes \mathbb{Z}(\mathbb{A}^1)$ embeds G as a summand of K. $\square$

For every presheaf F we define $\tilde{C}_m(F)$ to be the quotient presheaf $C_m(F)/F$. That is, $\tilde{C}_m(F)(U)$ is $F(U \times \mathbb{A}^m)/F(U)$. Thus we have split exact sequences $0 \to F \to C_m(F) \to \tilde{C}_m(F) \to 0$.

Corollary 12.18. *$\tilde{C}_m(F)$ is a direct summand of $\tilde{C}_m(F) \otimes I^1$ for all $m \geq 0$.*

Proof. It is easy to see that $F \to C_m F$ is a deformation retract, so 12.18 is a special case of 12.17. □

Proposition 12.19. *Let G be a Nisnevich sheaf on Sm/k such that $H^n_{Nis}(-,G)$ is homotopy invariant for all n. Then for all n and for all presheaves F, there is an isomorphism*

$$\operatorname{Hom}_{\mathbf{D}^-_{Nis}}((C_*F)_{Nis}, G[n]) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{D}^-_{Nis}}(F_{Nis}, G[n]).$$

Proof. Write $\operatorname{Ext}^n(C,G)$ for $\operatorname{Hom}_{\mathbf{D}^-_{Nis}}(C, G[n])$. For each complex C, lemma 12.15 implies that $\operatorname{Ext}^q(C \otimes_{Nis} I^1, G) = 0$ for all q. For $C = (\tilde{C}_p F)_{Nis}$, 12.18 yields $\operatorname{Ext}^q((\tilde{C}_p F)_{Nis}, G) = 0$.

Note that $\operatorname{Ext}^q(C,G) = H^q \mathbb{R}\operatorname{Hom}(C,G)$ for any C; see [Wei94, 10.7.4]. As in the proof of 10.10, a resolution $G \to I^*$ yields a first quadrant Hom double complex $\operatorname{Hom}((\tilde{C}_*F)_{Nis}, I^*)$ and hence a first quadrant spectral sequence

$$E_1^{p,q} = \operatorname{Ext}^q((\tilde{C}_p F)_{Nis}, G) \Rightarrow \operatorname{Ext}^{p+q}((\tilde{C}_* F)_{Nis}, G)$$

(see [Wei94, 5.6.1]). Since every $E_1^{p,q}$ vanishes, this implies that $\operatorname{Ext}^n((\tilde{C}_*F)_{Nis}, G) = 0$ for all n. In turn, this implies the conclusion of 12.19, viz., $\operatorname{Ext}^n((C_*F)_{Nis}, G) \cong \operatorname{Ext}^n(F, G)$ for all n. □

Lecture 13

Nisnevich sheaves with transfers

We now consider the category $Sh_{Nis}(Cor_k)$ of Nisnevich sheaves with transfers. As with étale sheaves, we say that a presheaf with transfers F is a **Nisnevich sheaf with transfers** if its underlying presheaf is a Nisnevich sheaf on Sm/k. Clearly, every étale sheaf with transfers is a Nisnevich sheaf with transfers.

Theorem 13.1. *Let F be a presheaf with transfers, and write F_{Nis} for the sheafification of the underlying presheaf. Then F_{Nis} has a unique structure of presheaf with transfers such that $F \to F_{Nis}$ is a morphism of presheaves with transfers.*

Consequently, $Sh_{Nis}(Cor_k)$ is an abelian category, and the forgetful functor $Sh_{Nis}(Cor_k) \hookrightarrow \mathbf{PST}(k)$ has a left adjoint $(F \mapsto F_{Nis})$ which is exact and commutes with the forgetful functor to (pre)sheaves on Sm/k.

Finally, $Sh_{Nis}(Cor_k)$ has enough injectives.

Proof. The Nisnevich analogue of 6.16, is valid; just replace 'étale cover' by 'Nisnevich cover' in the proof. As explained after 6.12, the Čech complex $\mathbb{Z}_{tr}(\check{U})$ is a Nisnevich resolution of $\mathbb{Z}_{tr}(X)$. With these two observations, the proofs of 6.17, 6.18, and 6.19 go through for the Nisnevich topology. $\square$

Exercise 13.2. By theorem 4.1, $\mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$ as complexes of Nisnevich sheaves with transfers. By 12.9, $\mathcal{O}^*/l = \mathcal{O}^* \otimes_{Nis} \mathbb{Z}/l$. Since $\mathbb{Z}/l(1) = \mathbb{Z}(1) \otimes^{\mathbb{L}}_{Nis} \mathbb{Z}/l$, it follows that there is a distinguished triangle of Nisnevich sheaves with transfers for each l:

$$\mu_l \to \mathbb{Z}/l(1) \to \mathcal{O}^*/l[-1] \to \mu_l[1].$$

Since $(\mathcal{O}^*/l)_{ét} = 0$, this recovers 4.8: $\mu_l \simeq \mathbb{Z}/l(1)_{ét}$.

Exercise 13.3. If F is a Nisnevich sheaf with transfers, modify example 6.20 to show that the sheaf $E^0(F)$ defined in 12.10 is a Nisnevich sheaf with transfers, and that the canonical flasque resolution $F \to E^*(F)$ is a complex of Nisnevich sheaves with transfers.

Lemma 13.4. *Let F be a Nisnevich sheaf with transfers. Then:*

1. *Its cohomology presheaves $H^n_{Nis}(-,F)$ are presheaves with transfers;*
2. *For any smooth X, we have $F(X) \cong \mathrm{Hom}_{Sh_{Nis}(Cor_k)}(\mathbb{Z}_{tr}(X), F)$;*
3. *For any smooth X and any $n \in \mathbb{Z}$,*

$$H^n_{Nis}(X,F) \cong Ext^n_{Sh_{Nis}(Cor_k)}(\mathbb{Z}_{tr}(X), F).$$

Proof. (Cf. 6.3, 6.21, and 6.23.) Assertion 2 is immediate from 13.1 and the Yoneda isomorphism $F(X) \cong \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{tr}(X), F)$. Now consider the canonical flasque resolution $F \to E^*(F)$ in $Sh_{Nis}(Sm/k)$. By 13.3, this is a resolution of sheaves with transfers. Since $H^n_{Nis}(-,F)$ is the cohomology of $E^*(F)$ as a presheaf, and hence as a presheaf with transfers, we get part 1.

For part 3, it suffices by part 2 to show that if F is an injective sheaf with transfers and $n > 0$, then $H^n_{Nis}(-,F) = 0$. Since $F \to E^0(F)$ must split in $Sh_{Nis}(Cor_k)$, $H^n_{Nis}(X,F)$ is a summand of $H^n_{Nis}(X, E^0(F)) = 0$, and must vanish. $\square$

Exercise 13.5. (Cf. 6.25.) Let K be any complex of Nisnevich sheaves of R-modules with transfer. Generalize 13.4 to show that its hyperext and hypercohomology agree in the sense that for any smooth X and $n \in \mathbb{Z}$:

$$\mathrm{Ext}^n(R_{tr}(X), K) \cong \mathbb{H}^n_{Nis}(X, K).$$

The following result allows us to bootstrap quasi-isomorphism results from the field level to the sheaf level.

Proposition 13.6. *Let $A \to B$ be a morphism of complexes of presheaves with transfers. Assume that their cohomology presheaves H^*A and H^*B are homotopy invariant, and that $A(\mathrm{Spec}\, E) \to B(\mathrm{Spec}\, E)$ is a quasi-isomorphism for every field E over k. Then $A_{Zar} \to B_{Zar}$ is a quasi-isomorphism in the Zariski topology.*

Proof. Let C be the mapping cone. By the 5-lemma, each H^iC is a homotopy invariant presheaf with transfers, which vanishes on $\operatorname{Spec} E$ for every field E over k. Corollary 11.2 states that $(H^iC)_{Zar} = 0$. This implies that C_{Zar} is acyclic as a complex of Zariski sheaves, i.e., that A_{Zar} and B_{Zar} are quasi-isomorphic in the Zariski topology. $\square$

The main result of this lecture, 13.11, as well as the next few lectures, depend upon the following result, whose proof will not be completed until 23.1. Theorem 13.7 allows us to bypass the notion of strictly $\mathbb{A}^1$-homotopy invariance (see 9.21) used in lecture 9. The case $n = 0$ of 13.7, that F_{Nis} is homotopy invariant, will be completed in 21.3.

Theorem 13.7. *Let k be a perfect field and F a homotopy invariant presheaf with transfers. Then each presheaf $H^n_{Nis}(-, F_{Nis})$ is homotopy invariant.*

The proofs of the following results are all based upon a combination of theorem 13.7, lemma 13.4, and proposition 13.6.

Proposition 13.8. *Let k be a perfect field. If F is a homotopy invariant Nisnevich sheaf with transfers, then for all n and all smooth X:*

$$H^n_{Zar}(X, F) \cong H^n_{Nis}(X, F).$$

We will prove in 21.15 that F_{Zar} is a presheaf with transfer. This would simplify the proof of 13.8.

Proof. For $n = 0$ we have $H^0_{Nis}(X, F) = H^0_{Zar}(X, F) = F(X)$ for every sheaf. By the Leray spectral sequence, it now suffices to prove that $H^n_{Nis}(S, F) = 0$ for all $n > 0$ when S is a local scheme. By 13.4 and 13.7, each $H^n_{Nis}(-, F)$ is a homotopy invariant presheaf with transfers. By 11.2, it suffices to show that $H^n_{Nis}(\operatorname{Spec} E, F) = 0$ for every field E over k. But fields are Hensel local rings, and as such have no higher cohomology, i.e., $H^n_{Nis}(\operatorname{Spec} E, -) = 0$ for $n > 0$. $\square$

Proposition 13.9. *Let C be a be bounded above complex of Nisnevich sheaves with transfer, whose cohomology sheaves are homotopy invariant. Then its Zariski and Nisnevich hypercohomology agree:*

$$\mathbb{H}^n_{Zar}(X, C) \cong \mathbb{H}^n_{Nis}(X, C) \qquad \textit{for all smooth } X \textit{ and for all } n.$$

Proof. We will proceed by descending induction on $n-p$, where $C^i = 0$ for $i > p$. If $\dim X = d$, then $\mathbb{H}^n_{Zar}(X, C) = \mathbb{H}^n_{Nis}(X, C) = 0$ for all $n > p + d$, because $\mathrm{cd}_{Zar}(X)$ and $\mathrm{cd}_{Nis}(X)$ are at most d. By 13.1, both the good Nisnevich truncation τC and the p^{th}-cohomology sheaf $H^p = (C/\tau C)_{Nis}$ are Nisnevich sheaves with transfers. Setting $m = n - p$, we have a diagram

$$\begin{tikzcd}
H^{m-1}_{Zar}(X, H^p) \arrow[r] \arrow[d, "\cong"] & H^n_{Zar}(X, \tau C) \arrow[r] \arrow[d, "\cong"] & H^n_{Zar}(X, C) \arrow[r] \arrow[d] & H^m_{Zar}(X, H^p) \arrow[r] \arrow[d, "\cong"] & H^{n+1}_{Zar}(X, \tau C) \arrow[d, "\cong"] \\
H^{m-1}_{Nis}(X, H^p) \arrow[r] & H^n_{Nis}(X, \tau C) \arrow[r] & H^n_{Nis}(X, C) \arrow[r] & H^m_{Nis}(X, H^p) \arrow[r] & H^{n+1}_{Nis}(X, \tau C).
\end{tikzcd}$$

The four outer verticals are isomorphisms, by induction and 13.8. The statement now follows from the 5-lemma. □

Example 13.10. The motivic complex $\mathbb{Z}(i)$ is bounded above, and has homotopy invariant cohomology by 2.18. If A is any abelian group, the same is true for $A(i) = A \otimes_{\mathbb{Z}} \mathbb{Z}(i)$. By 13.9, the motivic cohomology of a smooth X could be computed using Nisnevich hypercohomology:

$$H^{n,i}(X, A) = \mathbb{H}^n_{Zar}(X, A(i)) = \mathbb{H}^n_{Nis}(X, A(i)).$$

This is the definition of motivic cohomology used in [VSF00]. Note that the motivic cohomology groups $H^{n,i}(X, A)$ are presheaves with transfers by 13.5.

Theorem 13.11. *Let k be a perfect field and F a presheaf with transfers such that $F_{Nis} = 0$. Then $(C_*F)_{Nis} \simeq 0$ in the Nisnevich topology, and $(C_*F)_{Zar} \simeq 0$ in the Zariski topology.*

Proof. Let F be a presheaf with transfers such that $F_{Nis} = 0$. We will first prove that $(C_*F)_{Nis} \simeq 0$ or, equivalently, that the homology presheaves $H_i = H_i C_* F$ satisfy $(H_i)_{Nis} = 0$ for all i. For $i < 0$ this is trivial; $C_i F = 0$ implies that $H_i C_* F = 0$. Since $(H_0)_{Nis}$ is a quotient of $F_{Nis} = 0$, it is also true for $i = 0$.

We shall proceed by induction on i, so we assume that $(H_j)_{Nis} = 0$ for all $j < i$. That is, we assume that $\tau(C_*F)_{Nis} \simeq (C_*F)_{Nis}$, where $\tau(C_*F)_{Nis}$ denotes the subcomplex of $(C_*F)_{Nis}$ obtained by good truncation at level i:

$$\tau(C_*F)_{Nis} \;\text{ is }\; \cdots \to (C_{i+1}F)_{Nis} \to (C_iF)_{Nis} \to d(C_iF)_{Nis} \to 0.$$

There is a canonical morphism $\tau(C_*F)_{Nis} \to (H_i)_{Nis}[i]$ and hence a morphism $f : (C_*F)_{Nis} \to (H_i)_{Nis}[i]$ in the derived category $\mathbf{D}^-_{Nis}$. Since f induces an isomorphism on the ith homology sheaves, it suffices to prove that $f = 0$.

The presheaf with transfers H_i is homotopy invariant by 2.18, so by 13.1 and 13.7 the sheaf $G = (H_i)_{Nis}$ satisfies the hypothesis of 12.19. Since $F_{Nis} = 0$, 12.19 yields

$$\mathrm{Hom}_{\mathbf{D}^-_{Nis}}((C_*F)_{Nis}, (H_i)_{Nis}[i]) \cong \mathrm{Hom}_{\mathbf{D}^-_{Nis}}(F_{Nis}, (H_i)_{Nis}[i]) = 0.$$

Hence $f = 0$ in $\mathbf{D}^-_{Nis}$, and this implies that $(H_i)_{Nis} = 0$.

We can now prove that $C_*F_{Zar} \simeq 0$. Each cohomology presheaf $H^i = H^iC_*F$ is a homotopy invariant presheaf with transfers by 2.18. Since $(C_*F)_{Nis} \simeq 0$, we have $C_*(F)(\operatorname{Spec} E) \simeq 0$ for every finitely generated field extension E of k (and hence for every field over k). Indeed, E is $\mathcal{O}^h_{X,x}$ for the generic point of some smooth X. Now apply 13.6 to $C_*F \to 0$. □

Here is a stalkwise restatement of theorem 13.11.

Corollary 13.12. *Let k be a perfect field and F a presheaf with transfers so that $F(\operatorname{Spec} \mathcal{O}^h_{X,x}) = 0$ for all smooth X and all $x \in X$. Then $(C_*F)(\operatorname{Spec} \mathcal{O}_{X,x}) \simeq 0$ for all X and all $x \in X$.*

Corollary 13.13. *Let $f : C_1 \to C_2$ be a map of bounded above cochain complexes of presheaves with transfers. If f induces a quasi-isomorphism over all Hensel local rings $\operatorname{Spec} \mathcal{O}^h_{X,x}$, then $\mathrm{Tot}(C_*C_1) \to \mathrm{Tot}(C_*C_2)$ induces a quasi-isomorphism over all local rings.*

Proof. Let $K = \mathrm{cone}(f)$ denote the mapping cone of f. By assumption, each H^pK is a presheaf with transfers which vanishes on all Hensel local schemes, i.e., $K_{Nis} \simeq 0$. By 13.11, $C_*H^pK \simeq 0$ in the Zariski topology.

Since K is a bounded above cochain complex, the double complex $C_*(K)$ is bounded. Hence the usual spectral sequence of a double complex (see [Wei94, 5.6.2]) converges to $H_* \operatorname{Tot} C_*(K)$. Since $C_qK(X) = K(X \times \Delta^q)$ we have $H^pC_qK = C_qH^pK$ for all p and q, and we have seen that each $H^qC_*H^pK$ vanishes on every local scheme X. The resulting collapse in the spectral sequence shows that $H_* \operatorname{Tot} C_*(K)$ vanishes on every local scheme, which yields the result. □

If $\mathcal{U} = \{U_1, \dots, U_n\}$ is a Zariski covering of X, we saw in 6.12 that the Čech complex

$$\mathbb{Z}_{tr}(\check{\mathcal{U}}) : \quad 0 \to \mathbb{Z}_{tr}(U_1 \cap \ldots \cap U_n) \to \ldots \to \oplus_i \mathbb{Z}_{tr}(U_i) \to 0$$

is a resolution of $\mathbb{Z}_{tr}(X)$ in the étale topology (and the Nisnevich topology). Surprisingly, this gets even better when we apply C_*.

Proposition 13.14. *If $\mathcal{U}$ is a Zariski covering of X then the Čech resolution* $\operatorname{Tot} C_*\mathbb{Z}_{tr}(\check{\mathcal{U}}) \to C_*\mathbb{Z}_{tr}(X)$ *is a quasi-isomorphism in the Zariski topology.*

Proof. Apply 13.13 to 6.14. □

Example 13.15. Applying 13.14 to the usual cover of $\mathbb{P}^1$ (by $\mathbb{P}^1 - \{0\}$ and $\mathbb{P}^1 - \{\infty\}$) allows us to deduce that $C_*\left(\mathbb{Z}_{tr}(\mathbb{P}^1)/\mathbb{Z}\right) \simeq C_*\mathbb{Z}_{tr}(\mathbb{G}_m)[1] = \mathbb{Z}(1)[2]$ for the Zariski topology, because $C_*\mathbb{Z}_{tr}(\mathbb{A}^1)/\mathbb{Z} \simeq 0$ by 2.23. This was already observed in example 6.15 for the étale topology. This example will be generalized in theorem 15.2 below.

Lecture 14

The category of motives

In this lecture, we define the triangulated category of (effective) motives over k, and the motive of a scheme in this category. The construction of $\mathbf{DM}^{\text{eff},-}_{Nis}(k,R)$ is parallel to the construction of $\mathbf{DM}^{\text{eff},-}_{ét}(k,R)$ in 9.2, and we will see that the categories are equivalent if $\mathbb{Q} \subseteq R$.

Write $\mathbf{D}^-$ for $\mathbf{D}^- Sh_{Nis}(Cor_k, R)$, and let $\mathcal{E}_{\mathbb{A}}$ denote the smallest thick subcategory of $\mathbf{D}^-$ containing every $R_{tr}(X \times \mathbb{A}^1) \to R_{tr}(X)$ and closed under direct sums. (See 9.1 and 9.2.) The quotient $\mathbf{D}^-/\mathcal{E}_{\mathbb{A}}$ is the localization $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$, where $W_{\mathbb{A}} = W_{\mathcal{E}_{\mathbb{A}}}$ is the class of maps in $\mathbf{D}^-$ whose cone is in $\mathcal{E}_{\mathbb{A}}$. A map in $W_{\mathbb{A}}$ is called an $\mathbb{A}^1$-*weak equivalence.*

Definition 14.1. The triangulated category of motives over k is defined to be the localization $\mathbf{DM}^{\text{eff},-}_{Nis}(k,R) = \mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ of $\mathbf{D}^- = \mathbf{D}^- Sh_{Nis}(Cor_k, R)$. (Cf. 9.2.) If X is a smooth scheme over k, we write $M(X)$ for the class of $\mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{\text{eff},-}_{Nis}(k,\mathbb{Z})$ and call it the *motive* of X.

In 8.17, we showed that the derived category $\mathbf{D}^-(Sh_{ét}(Cor_k, R))$ is a tensor triangulated category. The same argument works in the Nisnevich topology for $\mathbf{D}^- Sh_{Nis}(Cor_k, R)$. Here are the details.

Definition 14.2. If C and D are bounded above complexes of presheaves with transfers, we write $C \otimes^{tr}_{L,Nis} D$ for $(C \otimes^{tr}_L D)_{Nis}$. Because 6.12 holds for the Nisnevich topology, the Nisnevich analogues of 8.14, 8.15, 8.16, 8.17, and 8.18 hold. In particular, the derived category $\mathbf{D}^-$ of bounded above complexes of Nisnevich sheaves with transfer is a tensor triangulated category under $\otimes^{tr}_{L,Nis}$.

Given 14.2, the proofs of 9.4 and 9.5 go through to show that the tensor $\otimes^{tr}_{L,Nis}$ on $\mathbf{D}^-$ also endows the localization $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$ of $\mathbf{D}^- Sh_{Nis}(Cor_k, R)$ with the structure of a tensor triangulated category.

Our definitions of $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$ and $M(X)$ are equivalent to the definitions in [TriCa, p. 205]. This follows by comparing the definition in *loc. cit.*, to theorem 14.10 below, using the following lemma.

Lemma 14.3. *For every bounded above complex K of sheaves of R-modules with transfer, the morphism $K \to \operatorname{Tot} C_*(K)$ is an $\mathbb{A}^1$-weak equivalence. Hence $K \cong \operatorname{Tot} C_*(K)$ in $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$.*

Proof. The proof of lemmas 9.11 and 9.14 go through in this setting. ☐

Remark 14.4. As is 9.16, we say that an object L of $\mathbf{D}^-$ is called $\mathbb{A}^1$-*local* (for the Nisnevich topology) if $\operatorname{Hom}_{\mathbf{D}^-}(-,L)$ sends $\mathbb{A}^1$-weak equivalences to isomorphisms. The proof of 9.20 goes through in the Nisnevich setting to show that if Y is $\mathbb{A}^1$-local then

$$\operatorname{Hom}_{\mathbf{DM}^{\mathrm{eff},-}_{Nis}}(X,Y) \cong \operatorname{Hom}_{\mathbf{D}^-}(X,Y).$$

Let F be a Nisnevich sheaf with transfers. Then F is $\mathbb{A}^1$-local if and only if F is homotopy invariant, because the proof of 9.23 goes through using 13.4 and 13.7. This is the easy case of the following proposition.

Proposition 14.5. *Let k be a perfect field and K a bounded above cochain complex of Nisnevich sheaves of R-modules with transfer. Then K is $\mathbb{A}^1$-local if and only if the sheaves $a_{Nis}(H^nK)$ are all homotopy invariant.*

Proof. Suppose first that the cohomology sheaves of K are homotopy invariant. By 13.7 applied to $F = a_{Nis}(H^qK)$, the presheaves $H^n_{Nis}(-,F)$ are homotopy invariant. As in the proof of 9.23, this implies that each $a_{Nis}(H^qK)$ is $\mathbb{A}^1$-local. Because $cd_{Nis}(X) < \infty$, the hyperext spectral sequence (see [Wei94, 5.7.9])

$$E_2^{pq}(X) = \operatorname{Ext}^p(R_{tr}(X), a_{Nis}H^qK) \Longrightarrow \operatorname{Hom}_{\mathbf{D}^-}(R_{tr}(X), K[p+q])$$

is bounded and converges. The map f induces a morphism from it to the corresponding spectral sequence for $X \times \mathbb{A}^1$. By the Comparison Theorem ([Wei94, 5.2.12]), f induces an isomorphism from $\operatorname{Hom}_{\mathbf{D}^-}(R_{tr}(X)[n],K)$ to $\operatorname{Hom}_{\mathbf{D}^-}(R_{tr}(X\times\mathbb{A}^1)[n],K)$ for each n. By 9.17, K is $\mathbb{A}^1$-local.

Now suppose that K is $\mathbb{A}^1$-local. The cohomology presheaves of $K' = \operatorname{Tot} C_*(K)$ are homotopy invariant by 2.18. Theorem 13.7 applied to the cohomology presheaves $H^q K'$ shows that the sheaves $a_{Nis}(H^q K')$ are homotopy invariant. The first part of this proof shows that K' is $\mathbb{A}^1$-local. By lemma 14.3, the canonical map $K \to K'$ is an $\mathbb{A}^1$-weak equivalence. By 9.18, which goes through for the Nisnevich topology, $K \to K'$ is an isomorphism in $\mathbf{D}^-$. Hence the sheaves $a_{Nis}(H^n K) \cong a_{Nis}(H^n K')$ are homotopy invariant. $\square$

Corollary 14.6. *Let k be a perfect field and K a bounded above cochain complex of Nisnevich sheaves of R-modules with transfer. If the presheaves $H^n(K)$ are all homotopy invariant, then K is $\mathbb{A}^1$-local.*

Proof. Combine 13.7 and 14.5. $\square$

Example 14.7. Here is an example to show that the converse does not hold in 14.6. Consider the complex K of example 6.15:

$$0 \to \mathbb{Z}_{tr}(\mathbb{G}_m) \to 2\mathbb{Z}_{tr}(\mathbb{A}^1, 1) \to \mathbb{Z}_{tr}(\mathbb{P}^1, 1) \to 0.$$

Evaluating at $\operatorname{Spec}(k)$ and at $\mathbb{A}^1$, it is easy to see that the cohomology presheaf $H^2 K$ is not homotopy invariant (consider an embedding of $\mathbb{A}^1$ in $\mathbb{P}^1$ whose image contains both 0 and 1). On the other hand K is $\mathbb{A}^1$-local, because its cohomology sheaves $a_{Nis} H^*(K)$ vanish by 6.14.

Exercise 14.8 (Mayer-Vietoris). For each open cover $\{U, V\}$ of a smooth scheme X, show that the "Mayer-Vietoris" triangle

$$\mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(U \cap V)[1]$$

is distinguished in $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k, R)$. *Hint:* Use a Nisnevich analogue of 6.14 to show that the associated Čech complex is isomorphic to zero in $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k, R)$. Cf. [TriCa, 3.2.6].

Definition 14.9. Let $\mathcal{L}_{Nis}$ denote the full subcategory of $\mathbf{D}^-$ consisting of complexes with homotopy invariant cohomology sheaves. By 14.5, it is also the category of $\mathbb{A}^1$-local complexes. If E and F are in $\mathcal{L}$, then we define $E \otimes_{\mathcal{L}} F = \operatorname{Tot} C_*(E \otimes^{tr}_{L,Nis} F)$. By 2.18 and 14.6, $E \otimes_{\mathcal{L}} F$ is $\mathbb{A}^1$-local.

Theorem 14.10. *The category $(\mathcal{L}_{Nis}, \otimes_{\mathcal{L}})$ is a tensor triangulated category, and the canonical functor*

$$\mathcal{L}_{Nis} \to \mathbf{D}^-[W_{\mathbb{A}}^{-1}] = \mathbf{DM}^{\mathrm{eff},-}_{Nis}(k, R)$$

is an equivalence of tensor triangulated categories.

Proof. The category $\mathcal{L} = \mathcal{L}_{Nis}$ is a thick subcategory of $\mathbf{D}^-$. By 14.4 and 14.5, the functor $\mathcal{L} \to \mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ is fully faithful. By 14.3, every object K of $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ is isomorphic to $\operatorname{Tot} C_*(K)$. By 13.7, $\operatorname{Tot} C_*(K)$ is in $\mathcal{L}$. Hence $\mathcal{L}$ is equivalent to $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$.

It follows that $\mathcal{L}$ is a tensor triangulated category, because $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$ is. If E and F are $\mathbb{A}^1$-local, we have seen that the tensor product $E \otimes_{L,Nis}^{tr} F$ is naturally isomorphic to $E \otimes_{\mathcal{L}} F$ in $\mathbf{D}^-[W_{\mathbb{A}}^{-1}]$. That is, $\otimes_{\mathcal{L}}$ is isomorphic to the induced tensor operation on $\mathcal{L}$. □

In [TriCa, p. 210], the tensor structure on $\mathcal{L}_{Nis}$ was defined using $\otimes_{\mathcal{L}}$.

Next, recall from 9.7 that two parallel morphisms f and g of sheaves are said to be $\mathbb{A}^1$-*homotopic* if there is a map $F \otimes_L^{tr} \mathbb{Z}_{tr}(\mathbb{A}^1) \to G$ whose restrictions along 0 and 1 coincide with f and g, respectively. The proof of 9.9 shows that $\mathbb{A}^1$-homotopic morphisms between Nisnevich sheaves became equal in $\mathbf{DM}_{Nis}^{\mathrm{eff},-}(k,R)$.

Proposition 14.11. *Let C and D be bounded above complexes of Nisnevich sheaves with transfer, whose cohomology sheaves are homotopy invariant. If C and D are $\mathbb{A}^1$-local, then $\mathbb{A}^1$-homotopic maps $f, g : C \to D$ induce the same maps on hypercohomology:*

$$f = g : \mathbb{H}^*_{Zar}(X, C) \to \mathbb{H}^*_{Zar}(X, D).$$

Proof. To prove the proposition, write $\mathbf{DM}$ for $\mathbf{DM}_{Nis}^{\mathrm{eff},-}(k)$. Combining 13.9 with 13.5, we see that

$$H^n_{Zar}(X, C) \cong \operatorname{Hom}_{\mathbf{D}^-}(\mathbb{Z}_{tr}(X), C[n]).$$

If C is $\mathbb{A}^1$-local, this equals $\operatorname{Hom}_{\mathbf{DM}}(\mathbb{Z}_{tr}(X), C[n])$ by 14.4. Since f and g agree in $\mathbf{DM}$, they induce the same map from $H^n(X, C) \cong \operatorname{Hom}_{\mathbf{DM}}(\mathbb{Z}_{tr}(X), C[n])$ to $H^n(X, D) \cong \operatorname{Hom}_{\mathbf{DM}}(\mathbb{Z}_{tr}(X), D[n])$, as asserted. □

We will need the following elementary result for $R = \mathbb{Q}$ in 14.22 below. It is proven by replacing $\mathbb{Z}(X)$ by $R_{tr}(X)$ in the proof of 12.13.

Lemma 14.12. *The smallest class in $\mathbf{D}^-$ which contains all the $R_{tr}(X)$ and is closed under quasi-isomorphisms, direct sums, shifts, and cones is all of $\mathbf{D}^-$.*

We now consider the case when the coefficient ring R contains $\mathbb{Q}$. Our first goal is to identify étale and Nisnevich motivic cohomology (14.16). We will then describe $\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k,R)$ (in 14.19), and finally show that $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R) \cong \mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}(k,R)$ (14.22).

Lemma 14.13. *Let F be a homotopy invariant Zariski sheaf of $\mathbb{Q}$-modules with transfers. Then F is also an étale sheaf with transfers.*

Proof. It suffices to show that the presheaf kernel and cokernel of $F \to F_{\acute{e}t}$ vanish. By 6.17, these are presheaves with transfers. Thus we may suppose that $F_{\acute{e}t} = 0$. If $F \neq 0$ then there is a point $x \in X$ and a nonzero element $c \in F(S)$, $S = \operatorname{Spec} \mathcal{O}_{X,x}$. Since $F_{\acute{e}t} = 0$, there is a finite étale map $S' \to S$ with $c|_{S'} = 0$. As in 1.11, the composition of the transfers and inclusion

$$F(S) \to F(S') \to F(S)$$

is multiplication by d, the degree of $S' \to S$. Hence this composition is an isomorphism. Since it sends c to zero, we have $c = 0$. This contradiction shows that $F = 0$, as desired. $\square$

Corollary 14.14. *If F is an homotopy invariant presheaf of $\mathbb{Q}$-modules with transfer, then $F_{Nis} = F_{\acute{e}t}$.*

Proof. By 13.7, F_{Nis} is homotopy invariant, so 14.13 applies. $\square$

Proposition 14.15. *If F is an étale sheaf of $\mathbb{Q}$-modules, then*

$$H^n_{\acute{e}t}(-,F) = H^n_{Nis}(-,F).$$

Proof. We need to prove that $H^n_{\acute{e}t}(S,F) = 0$ for $n > 0$ when S is Hensel local. Given this, the result will follow from the Leray spectral sequence. Pick $c \in H^n_{\acute{e}t}(S,F)$. Then there is a finite étale map $S' \to S$ with $c|_{S'} = 0$. We may assume that S' is connected, of degree d. The composition of the étale trace map and the inclusion

$$H^n_{\acute{e}t}(S,F) \longrightarrow H^n_{\acute{e}t}(S',F) \longrightarrow H^n_{\acute{e}t}(S,F)$$

is multiplication by d (see [Mil80, V.1.12]) and so an isomorphism. Since it sends c to zero, we have $c = 0$. Hence $H^n(S,F) = 0$. $\square$

Recall from 10.1 that the étale (or Lichtenbaum) motivic cohomology $H^{p,q}_L(X,\mathbb{Q})$ is defined to be the étale hypercohomology of the complex $\mathbb{Q}(q)$.

Theorem 14.16. *Let k be a perfect field. If C is a complex of presheaves of $\mathbb{Q}$-modules with transfer with homotopy invariant cohomology, then*

$$\mathbb{H}^*_{\acute{e}t}(X, C_{\acute{e}t}) = \mathbb{H}^*_{Nis}(X, C_{Nis})$$

for every X in Sm/k. In particular

$$H^{p,q}_L(X, \mathbb{Q}) = H^{p,q}(X, \mathbb{Q}).$$

Proof. Consider $F = H^q C$. By 14.14, $F_{Nis} = F_{\acute{e}t}$. By 14.15, we have isomorphisms $H^i_{\acute{e}t}(X, F_{\acute{e}t}) \to H^i_{Nis}(X, F_{Nis})$. Comparing the hypercohomology spectral sequences for the Nisnevich and the étale topology yields the result.

In particular, the result applies to $C = \mathbb{Q}(q)$ by 13.10. □

Lemma 14.17. *Let F be any homotopy invariant étale sheaf of $\mathbb{Q}$-modules with transfers. Then F is strictly $\mathbb{A}^1$-homotopy invariant in the sense of 9.21.*

Proof. By 14.15 and 13.7, we have $H^p_{\acute{e}t}(X, F) \cong H^p_{\acute{e}t}(X \times \mathbb{A}^1, F)$ for all smooth X, i.e., F is strictly homotopy invariant. □

For clarity, let us say that a complex K is *étale $\mathbb{A}^1$-local* if it is $\mathbb{A}^1$-local for the étale topology (as in 9.16), and *Nisnevich $\mathbb{A}^1$-local* if it is $\mathbb{A}^1$-local for the Nisnevich topology (as in 14.4).

We will write $\mathbf{D}^-_{\acute{e}t}$ for $\mathbf{D}^-(Sh_{\acute{e}t}(Cor_k, R))$.

Proposition 14.18. *Let k be a perfect field and suppose that $\mathbb{Q} \subseteq R$. If K is a bounded above cochain complex of étale sheaves of R-modules with transfer, then K is étale $\mathbb{A}^1$-local if and only if the sheaves $a_{\acute{e}t}(H^n K)$ are homotopy invariant.*

In particular, each $\mathbb{Q}(j)$ is an étale $\mathbb{A}^1$-local complex.

Proof. Suppose first that the sheaves $a_{\acute{e}t} H^n(K)$ are homotopy invariant. By 14.17 and 9.23, they are étale $\mathbb{A}^1$-local. Since $\mathbb{Q} \subseteq R$, we have $\mathrm{cd}_R(k) = 0$, so K is étale $\mathbb{A}^1$-local by 9.28.

Conversely, suppose that K is étale $\mathbb{A}^1$-local and set $K' = \operatorname{Tot} C_*(K)$. By 2.18, each $H^n(K')$ is $\mathbb{A}^1$-homotopy invariant. By theorem 13.7 and 14.13, each étale sheaves $a_{\acute{e}t}(H^n K')$ is homotopy invariant. The first part of this proof shows that K' is étale $\mathbb{A}^1$-local. By 9.14, $K \to K'$ is an (étale) $\mathbb{A}^1$-weak equivalence, and an isomorphism in $\mathbf{DM}^{\text{eff},-}_{\acute{e}t}(k, R)$. By 9.18, $K \to K'$ is an isomorphism in $\mathbf{D}^-_{\acute{e}t}$. Hence each sheaf $a_{\acute{e}t}(H^n K)$ is isomorphic to $a_{\acute{e}t}(H^n K')$, and is therefore also homotopy invariant. □

Let $\mathcal{L}_{ét}$ denote the full subcategory of $\mathbf{D}^-_{ét}$ consisting of complexes with homotopy invariant cohomology sheaves. By 14.18, it is also the subcategory of étale $\mathbb{A}^1$-local complexes.

Theorem 14.19. *The natural functor $\mathcal{L}_{ét} \to \mathbf{DM}^{\text{eff},-}_{ét}(k,R)$ is an equivalence of triangulated categories if $\mathbb{Q} \subseteq R$.*

Proof. The functor is full and faithful by 9.20 and 14.18. Since every K in $\mathbf{D}^-_{ét}$ becomes isomorphic to $\operatorname{Tot} C_*(K)$ in $\mathbf{DM}^{\text{eff},-}_{ét}$ by 9.14, and $\operatorname{Tot} C_*(K)$ is in $\mathcal{L}_{ét}$ by 2.18, the functor is an equivalence. $\square$

Remark 14.20. Theorem 14.19 implies that $\mathcal{L}_{ét}$ is a tensor triangulated category. As in the proof of 9.32 and 14.10, 14.3 and 14.6 show that the tensor operation of $\mathcal{L}_{ét}$ is isomorphic to the operation $\otimes_{\mathcal{L}}$ defined in 9.31.

Corollary 14.21. *For every smooth X, the étale motivic cohomology groups $H^{p,q}_L(X,\mathbb{Q})$ may be computed in $\mathbf{DM}^-_{ét}$:*

$$H^{p,q}_L(X,\mathbb{Q}) \cong \operatorname{Hom}_{\mathbf{DM}^-_{ét}}(\mathbb{Q}_{tr}(X), \mathbb{Q}(q)[p]).$$

Proof. Write $\mathbf{D}^-_{ét}$ for $\mathbf{D}^-(Sh_{et}(Cor_k,\mathbb{Q}))$ and write $\mathbf{DM}$ for $\mathbf{DM}^-_{et}(k,\mathbb{Q})$. By 9.20, we have

$$\operatorname{Hom}_{\mathbf{DM}}(\mathbb{Q}_{tr}(X), \mathbb{Q}(q)[p]) = \operatorname{Hom}_{\mathbf{D}^-_{ét}}(\mathbb{Q}_{tr}(X), \mathbb{Q}(q)[p]) = \operatorname{Ext}^p(\mathbb{Q}_{tr}(X), \mathbb{Q}(q)).$$

By 6.25, this Ext group is $H^{p,q}_L(X,\mathbb{Q}) = H^p_{ét}(X,\mathbb{Q}(q))$. $\square$

Theorem 14.22. *If $\mathbb{Q} \subseteq R$, then $\mathbf{DM}^{\text{eff},-}_{Nis}(k,R) \to \mathbf{DM}^{\text{eff},-}_{ét}(k,R)$ is an equivalence of tensor triangulated categories.*

Proof. Because sheafification is exact, it induces a triangulated functor from $\mathbf{D}^-_{Nis} = \mathbf{D}^-(Sh_{Nis}(Cor_k,R))$ to $\mathbf{D}^-_{ét} = \mathbf{D}^-(Sh_{ét}(Cor_k,R))$. By definitions 9.14 and 14.2, we have $(K \otimes^{tr}_{L,Nis} L)_{ét} = K \otimes^{tr}_{L,ét} L$. Comparing definitions, we see that $\mathbf{D}^-_{Nis} \to \mathbf{D}^-_{ét}$ sends Nisnevich $\mathbb{A}^1$-weak equivalences to étale $\mathbb{A}^1$-weak equivalences, so it induces a tensor triangulated functor σ from $\mathbf{DM}^{\text{eff},-}_{Nis}(k,R)$ to $\mathbf{DM}^{\text{eff},-}_{ét}(k,R)$. Clearly, $\mathbf{D}^-_{Nis} \to \mathbf{D}^-_{ét}$ and $\mathbf{DM}^{\text{eff},-}_{Nis} \to \mathbf{DM}^{\text{eff},-}_{ét}$ are onto on objects.

It remains to show that the functor σ is full and faithful, i.e., that we have $\operatorname{Hom}_{\mathbf{DM}^{\text{eff},-}_{Nis}}(K,L) \cong \operatorname{Hom}_{\mathbf{DM}^{\text{eff},-}_{ét}}(K_{ét},L_{ét})$. By theorem 14.10, we may assume that L is in $\mathcal{L}_{Nis}$. The class of objects K so that $\operatorname{Hom}_{\mathbf{DM}^{\text{eff},-}_{Nis}}(K,L[n]) \cong$

$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}}(K_{\acute{e}t}, L_{\acute{e}t}[n])$ for all n is closed under quasi-isomorphisms, direct sums, shifts, and cones. By 14.12, it suffices to show that each $R_{tr}(X)$ is in this class. But then by 14.4, 13.5, 9.20, and 6.25, we have

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}_{Nis}}(R_{tr}(X), L[n]) \cong \mathrm{Hom}_{\mathbf{D}^{-}_{Nis}}(R_{tr}(X), L[n]) \cong H^n_{Nis}(X, L)$$

and

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}_{\acute{e}t}}(R_{tr}(X), L_{\acute{e}t}[n]) \cong \mathrm{Hom}_{\mathbf{D}^{-}_{\acute{e}t}}(R_{tr}(X), L_{\acute{e}t}[n]) \cong H^n_{\acute{e}t}(X, L_{\acute{e}t}).$$

These groups are isomorphic by 14.16, as required. $\square$

Lecture 15

The complex $\mathbb{Z}(n)$ and $\mathbb{P}^n$

The goal of this lecture is to interpret the motivic complex $\mathbb{Z}(n)$ in terms of $\mathbb{Z}_{tr}(\mathbb{P}^n)$ and use this to show that the product on motivic cohomology is graded-commutative. We begin by observing that $M(\mathbb{P}^n - \{0\}) \cong M(\mathbb{P}^{n-1})$.

Lemma 15.1. *There is a chain homotopy equivalence:*

$$C_*\mathbb{Z}_{tr}(\mathbb{P}^n - \{0\}) \simeq C_*\mathbb{Z}_{tr}(\mathbb{P}^{n-1}).$$

Proof. Consider the projection $(\mathbb{P}^n - 0) \to \mathbb{P}^{n-1}$ sending $(x_0 : \cdots : x_n)$ to $(x_1 : \cdots : x_n)$, where 0 is $(1 : 0 : \cdots : 0)$. This map has affine fibers. The self homotopy $\lambda(x_0 : \cdots : x_n) \to (\lambda x_0 : x_1 : \cdots : x_n)$ is well defined on $\mathbb{P}^n - \{0\} \times \mathbb{A}^1$, even for $\lambda = 0$, because one of $x_1, \ldots, x_n$ is always non zero. Hence the projection and the section $(x_1 : \cdots : x_n) \mapsto (0 : x_1 : \cdots : x_n)$ are inverse $\mathbb{A}^1$-homotopy equivalences. The lemma now follows from 2.25. $\square$

Theorem 15.2. *If k is a perfect field, there is a quasi-isomorphism of Zariski sheaves for each n:*

$$C_*\left(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1})\right) \simeq C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})[n] = \mathbb{Z}(n)[2n].$$

In particular, $C_*\left(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1})\right)(X) \simeq \mathbb{Z}(n)[2n](X)$ for any smooth local X.

Our proof will use theorem 13.11, whose proof depended upon theorem 13.7, a result whose proof we have postponed until lecture 23. The requirement that k be perfect is only needed for 13.7 (and hence 13.11).

Proof. Let $\mathcal{U}$ be the usual cover of $\mathbb{P}^n$ by $(n+1)$ copies of $\mathbb{A}^n$ and note that n of these form a cover $\mathcal{V}$ of $\mathbb{P}^n - \{0\}$. The intersection of $i+1$ of these $\mathbb{A}^n$ is $\mathbb{A}^{n-i} \times (\mathbb{A}^1 - \{0\})^i$. By 6.14, we have quasi-isomorphisms $\mathbb{Z}_{tr}(\check{\mathcal{U}}) \to \mathbb{Z}_{tr}(\mathbb{P}^n)$ and $\mathbb{Z}_{tr}(\check{\mathcal{V}}) \to \mathbb{Z}_{tr}(\mathbb{P}^n - 0)$ of complexes of Nisnevich sheaves with transfers. Hence the quotient complex $Q_* = \mathbb{Z}_{tr}(\check{\mathcal{U}})/\mathbb{Z}_{tr}(\check{\mathcal{V}})$ is a resolution of $\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^n - 0)$ as a Nisnevich sheaf. By 13.13 and 15.1, or by 13.14, $\operatorname{Tot} C_* Q_*$ is quasi-isomorphic to $C_*\left(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^n - 0)\right)$ and hence to $C_*(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1}))$ for the Zariski topology.

On the other hand, we know from 2.12 that for $T = \mathbb{A}^1 - 0$ the sequence

$$\begin{aligned} 0 \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) \to \mathbb{Z}_{tr}(T^n) \to \oplus_i \mathbb{Z}_{tr}(T^{n-1}) \to \oplus_{i,j} \mathbb{Z}_{tr}(T^{n-2}) \to \\ \cdots \to \oplus_{i,j} \mathbb{Z}_{tr}(T^2) \to \oplus_i \mathbb{Z}_{tr}(T) \to \mathbb{Z} \to 0 \end{aligned}$$

is split exact. Rewriting this as $0 \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) \to R_n \to R_{n-1} \to \cdots \to R_0 \to 0$, with $R_n = \mathbb{Z}_{tr}(T^n)$, $R_{n-1} = \oplus_i \mathbb{Z}_{tr}(T^{n-1})$, and $R_0 = \mathbb{Z}$, we may regard it as a chain homotopy equivalence $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})[n] \to R_*$. With this indexing there is a natural map $Q_* \to R_*$ whose typical term is a direct sum of the projections

$$\mathbb{Z}_{tr}(\mathbb{A}^{n-i} \times T^i) \to \mathbb{Z}_{tr}(T^i).$$

These are $\mathbb{A}^1$-homotopy equivalences (see 2.24). Applying C_* turns them into quasi-isomorphisms by 2.25. Hence we have quasi-isomorphisms of total complexes of presheaves with transfers

$$\operatorname{Tot} C_* Q_* \xrightarrow{\simeq} \operatorname{Tot} C_* R_* \xleftarrow{\simeq} C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})[n].$$

Combining with $\operatorname{Tot} C_* Q_* \simeq C_*\left(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1})\right)$ yields the result in the Zariski topology. $\square$

If $n = 1$, it is easy to see that the isomorphisms of 13.15 and 15.2 agree. Figure 15.1 illustrates the proof of theorem 15.2 when $n = 2$. We have written 'X' for $C_*\mathbb{Z}_{tr}(X)$ in order to save space, and '$\mathbb{A}^1$−h.e.' for $\mathbb{A}^1$-homotopy equivalence.

Corollary 15.3. *For each n there is a quasi-isomorphism for the Zariski topology*

$$C_*\left(\mathbb{Z}_{tr}(\mathbb{A}^n - 0)/\mathbb{Z}\right) \simeq \mathbb{Z}(n)[2n-1].$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \times (\mathbb{A}^1 - 0) \arrow[r] \arrow[d] & 2\mathbb{A}^2 \arrow[r] \arrow[d] & \mathbb{P}^2 - 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & (\mathbb{A}^1 - 0)^2 \arrow[r] \arrow[d, "="] & 3\left(\mathbb{A}^1 \times (\mathbb{A}^1 - 0)\right) \arrow[r] \arrow[d] & 3\mathbb{A}^2 \arrow[r] \arrow[d] & \mathbb{P}^2 \arrow[d] \\
0 \arrow[r] \arrow[d] & (\mathbb{A}^1 - 0)^2 \arrow[r] \arrow[d, "="] & 2\left(\mathbb{A}^1 \times (\mathbb{A}^1 - 0)\right) \arrow[r] \arrow[d, "\mathbb{A}^1\text{--h.e.}"] & \mathbb{A}^2 \arrow[r] \arrow[d, "\mathbb{A}^1\text{--h.e.}"] & \mathbb{P}^2/(\mathbb{P}^2 - 0) \arrow[d] \\
\mathbb{G}_m^{\wedge 2} \arrow[r] & (\mathbb{A}^1 - 0)^2 \arrow[r] & 2(\mathbb{A}^1 - 0) \arrow[r] & \mathrm{pt} \arrow[r] & 0
\end{tikzcd}$$

Figure 15.1: The case $n = 2$ of theorem 15.2

Proof. Applying 13.14 and 15.1 to the cover of $\mathbb{P}^n$ by $\mathbb{A}^n$ and $\mathbb{P}^n - 0$, we see that the sequence

$$0 \to C_*\mathbb{Z}_{tr}(\mathbb{A}^n - 0) \to C_*\mathbb{Z}_{tr}(\mathbb{A}^n) \oplus C_*\mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \to C_*\mathbb{Z}_{tr}(\mathbb{P}^n) \to 0$$

becomes exact for the Zariski topology. The result now follows from theorem 15.2, since $C_*\mathbb{Z}_{tr}(\mathbb{A}^n) \simeq C_*\mathbb{Z}_{tr}(\operatorname{Spec} k) \simeq \mathbb{Z}$ by 2.23 and 2.13. $\square$

Exercise 15.4. Show that the map $C_*\mathbb{Z}_{tr}(\mathbb{P}^i) \to \mathbb{Z}(i)[2i]$ of theorem 15.2 factors through the natural inclusion $C_*\mathbb{Z}_{tr}(\mathbb{P}^i) \to C_*\mathbb{Z}_{tr}(\mathbb{P}^n)$ for all $n > i$. *Hint*: First construct $\mathbb{Z}_{tr}(\check{\mathcal{U}}) \to \mathbb{Z}(1)[2]$ vanishing on $\mathbb{Z}_{tr}(U_0)$, and form

$$\mathbb{Z}_{tr}(\check{\mathcal{U}}) \xrightarrow{\Delta} \mathbb{Z}_{tr}(\check{\mathcal{U}}) \otimes \cdots \otimes \mathbb{Z}_{tr}(\check{\mathcal{U}}) \to \mathbb{Z}(1)[2]^{\otimes i} \to \mathbb{Z}(i)[2i].$$

Corollary 15.5. *There is a quasi-isomorphism*

$$M(\mathbb{P}^n) = C_*\mathbb{Z}_{tr}(\mathbb{P}^n) \xrightarrow{\simeq} \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \cdots \oplus \mathbb{Z}(n)[2n].$$

Proof. We proceed by induction, the case $n = 1$ being 13.15. By exercise 15.4, the maps $\mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \to \mathbb{Z}_{tr}(\mathbb{P}^n)$ is split injective in **DM**, because the quasi-isomorphism $\mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \to \oplus\mathbb{Z}(i)[i]$ factors through it. Hence the distinguished triangle

$$C_*\mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \longrightarrow C_*\mathbb{Z}_{tr}(\mathbb{P}^n) \longrightarrow \mathbb{Z}(n)[n] \longrightarrow C_*\mathbb{Z}_{tr}(\mathbb{P}^{n-1})[1]$$

splits. $\square$

Our re-interpretation of the motivic complexes allows us to show that the product in motivic cohomology is skew-commutative. This will be a consequence of the following construction, and some linear algebra.

Example 15.6. Consider the reflection automorphism τ of $\mathbb{P}^n$, $n \geq 1$, sending $(x_0 : x_1 : \cdots : x_n)$ to $(-x_0 : x_1 : \cdots : x_n)$. We claim that the induced automorphism of $C_*\mathbb{Z}_{tr}(\mathbb{P}^n)$ is $\mathbb{A}^1$-homotopic to the identity map, so that it is the identity map in $\mathbf{DM}^{\text{eff},-}_{Nis}$ (see 14.1 and 9.9).

To see this, consider the elementary correspondence from $\mathbb{P}^n \times \mathbb{A}^1$ (parametrized by $x_0, ..., x_n$ and t) to $\mathbb{P}^n$ (parametrized by $y_0, ..., y_n$) given by the subvariety Z of $\mathbb{P}^n \times \mathbb{A}^1 \times \mathbb{P}^n$ defined by the homogeneous equation(s)

$$y_i(x_0 y_i + t x_i y_0) = (t^2 - 1)x_i y_0^2, \qquad i = 1, \dots, n$$

together with $x_i y_j = x_j y_i$ for $1 \leq i, j \leq n$ if $n \geq 2$. (Exercise: check that this is an elementary correspondence!) The restrictions along $t = \pm 1$ yield two finite correspondences from $\mathbb{P}^n$ to itself, whose difference is $id_{\mathbb{P}^n} - \tau$.

Restricted to $\mathbb{P}^{n-1} \times \mathbb{A}^1$, this correspondence is the projection onto $\mathbb{P}^{n-1}$. Thus it induces an $\mathbb{A}^1$-homotopy between τ and the identity of $\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1})$. Applying C_*, we see from theorem 15.2 that it induces an $\mathbb{A}^1$-homotopy between the reflection automorphism τ of $\mathbb{Z}(n)[2n]$ and the identity, so that τ is the identity map in $\mathbf{DM}^{\text{eff},-}_{Nis}$.

The symmetric group Σ_n acts canonically on $\mathbb{A}^n$ by permuting coordinates. By inspection, this induces a Σ_n-action on the sheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$ and hence on the motivic complexes $\mathbb{Z}(n)$.

Proposition 15.7. *The action of the symmetric group Σ_n on $C_*\mathbb{Z}_{tr}(\mathbb{A}^n - 0)$ is $\mathbb{A}^1$-homotopic to the trivial action.*

Proof. Because the action is induced from an embedding $\Sigma_n \hookrightarrow GL_n(k)$, and every transposition acts as the reflection τ times an element of $SL_n(k)$, we see from example 15.6 that it suffices to show that the action of $SL_n(k)$ on $C_*\mathbb{Z}_{tr}(\mathbb{A}^n - 0)$ is chain homotopic to the trivial action.

Since every matrix in $SL_n(k)$ is a product of elementary matrices, it suffices to consider one elementary matrix $e_{ij}(a)$, $a \in k$. But multiplication by this matrix is $\mathbb{A}^1$-homotopic to the identity of $\mathbb{A}^n - 0$, by the homotopy $(x,t) \mapsto e_{ij}(at)x$ (see 9.8). In particular it is an $\mathbb{A}^1$-homotopy equivalence (see 2.24). By 2.25, the resulting endomorphism of $C_*\mathbb{Z}_{tr}(\mathbb{A}^n - 0)$ is chain homotopic to the identity. $\square$

Corollary 15.8. *The action of the symmetric group Σ_n on $\mathbb{Z}(n)$ is $\mathbb{A}^1$-homotopic to the trivial action. Hence it is trivial in $\mathbf{DM}^{\mathrm{eff},-}_{Nis}$, and on the motivic cohomology groups $H^{p,q}(X, \mathbb{Z}(n))$.*

Tensoring with a coefficient ring R does not affect the action, so it follows that Σ_n also acts trivially on $R(n)[2n]$, and on $H^{p,q}(X, R(n))$.

Proof. The action of Σ_n on $\mathbb{A}^n$ extends to an action on $\mathbb{P}^n$ fixing $\mathbb{P}^{n-1}$. In fact, all the constructions in the proof of theorem 15.2 and corollary 15.3 are equivariant. By 15.3, it suffices to show that the action of Σ_n on $C_*\mathbb{Z}_{tr}(\mathbb{A}^n - 0)$ is $\mathbb{A}^1$-homotopic to the trivial action. This follows from 15.7 and 14.11. $\square$

Recall from 3.10 that there is a pairing of presheaves $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i+j)$. By inspection of 3.9, this pairing is compatible with the action of the subgroup $\Sigma_i \times \Sigma_j$ of Σ_{i+j}, as well as with the permutation τ interchanging the first i and last j coordinates of $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge i+j})$.

Theorem 15.9. *The pairing defined in 3.11 is skew-commutative:*

$$H^p_{Zar}(X, \mathbb{Z}(i)) \otimes H^q_{Zar}(X, \mathbb{Z}(j)) \to H^{p+q}_{Zar}(X, \mathbb{Z}(i+j)).$$

Proof. As in 8A.2, the permutation τ fits into the commutative diagram

$$\begin{tikzcd}
H^p(X,\mathbb{Z}(i)) \otimes H^q(X,\mathbb{Z}(j)) \arrow[r] \arrow[d, "(-1)^{pq}"] & H^{p+q}(X,\mathbb{Z}(i)\otimes\mathbb{Z}(j)) \arrow[r] \arrow[d, "\tau"] & H^{p+q}(X,\mathbb{Z}(i+j)) \arrow[d, "\tau"] \\
H^q(X,\mathbb{Z}(j)) \otimes H^p(X,\mathbb{Z}(i)) \arrow[r] & H^{p+q}(X,\mathbb{Z}(j)\otimes\mathbb{Z}(i)) \arrow[r] & H^{p+q}(X,\mathbb{Z}(j+i))
\end{tikzcd}$$

and the right vertical map is the identity by proposition 15.8. $\square$

We conclude this lecture with a generalization of the decomposition 15.5 of $M(\mathbb{P}^n)$ to a projective bundle theorem.

Construction 15.10. Let $\mathbb{P} = \mathbb{P}(\mathcal{E}) \to X$ be a projective bundle associated to a vector bundle $\mathcal{E}$ of rank $n+1$. From 4.2, 13.10, and 13.5 we have an isomorphism

$$\mathrm{Pic}(\mathbb{P}) \cong H^2_{Nis}(\mathbb{P}, \mathbb{Z}(1)) \cong \mathrm{Hom}_{\mathbf{D}^-}(\mathbb{Z}_{tr}(\mathbb{P}), \mathbb{Z}(1)[2]).$$

Therefore the canonical line bundle yields a canonical map $\tau : \mathbb{Z}_{tr}(\mathbb{P}) \to \mathbb{Z}(1)[2]$ in $\mathbf{D}^-$. Recall from 10.4 that there are multiplication maps for all $i \geq 1$, from $\mathbb{Z}(1)^{\otimes^{tr} i} = \mathbb{Z}(1) \otimes^{tr} \cdots \otimes^{tr} \mathbb{Z}(1)$ to $\mathbb{Z}(i)$. For $i > 1$, we let τ^i denote the composite

$$\mathbb{Z}_{tr}(\mathbb{P}) \xrightarrow{\Delta} \mathbb{Z}_{tr}(\mathbb{P} \times \cdots \times \mathbb{P}) \xrightarrow{\cong} \mathbb{Z}_{tr}(\mathbb{P})^{\otimes^{tr} i} \xrightarrow{\tau^{\otimes i}} \mathbb{Z}(1)[2]^{\otimes^{tr} i} \longrightarrow \mathbb{Z}(i)[2i].$$

Finally, we extend the structure map $\sigma_0 : \mathbb{Z}_{tr}(\mathbb{P}) \to \mathbb{Z}_{tr}(X)$ to a canonical family of maps in $\mathbf{D}^-$

$$\sigma_i : \mathbb{Z}_{tr}(\mathbb{P}) \xrightarrow{\Delta} \mathbb{Z}_{tr}(\mathbb{P}) \otimes^{tr} \mathbb{Z}_{tr}(\mathbb{P}) \xrightarrow{\sigma_0 \otimes \tau^i} \mathbb{Z}_{tr}(X) \otimes \mathbb{Z}(i)[2i].$$

Exercise 15.11. Show that the canonical map in 15.10 induces the isomorphism $\mathbb{Z}_{tr}(\mathbb{P}^n_k) \cong \oplus_{i=0}^n \mathbb{Z}(i)[2i]$ of 15.5. *Hint*: Use exercise 15.4.

Theorem 15.12 (Projective Bundle Theorem). *Let $\mathbb{P}(\mathcal{E}) \to X$ be a projective bundle associated to a vector bundle $\mathcal{E}$ of rank $n+1$. Then the canonical map*

$$\oplus_{i=0}^n \mathbb{Z}_{tr}(X)(i)[2i] \to \mathbb{Z}_{tr}(\mathbb{P}(\mathcal{E}))$$

is an isomorphism in $\mathbf{DM}$.

Proof. Using induction on the number of open subsets in a trivialization of $\mathcal{E}$, together with the Mayer-Vietoris triangles 14.8, we are reduced to the case when $\mathbb{P}(\mathcal{E}) = X \times \mathbb{P}^n$. Since $\mathbb{Z}_{tr}(X \times \mathbb{P}^n) \cong \mathbb{Z}_{tr}(X) \otimes^{tr} \mathbb{Z}_{tr}(\mathbb{P}^n)$, we may even assume $X = \mathrm{Spec}(k)$. This case is given by exercise 15.11. $\square$

Lecture 16

Equidimensional cycles

In this lecture we introduce the notion of an equidimensional cycle, and use it to construct the Suslin-Friedlander chain complex $\mathbb{Z}^{SF}(i)$. We then show (in 16.7) that $\mathbb{Z}^{SF}(i)$ is quasi-isomorphic to $\mathbb{Z}(i)$. In lecture 19 (19.4) we shall compare $\mathbb{Z}^{SF}(i)$ to the complex defining higher Chow groups.

Let Z be a scheme of finite type over S such that every irreducible component of Z dominates a component of S. We say that Z is **equidimensional over** S of relative dimension m if for every point s of S, either Z_s is empty or each of its components have dimension m. If $S' \to S$ is any map, the pullback $S' \times_S Z$ is equidimensional over S' of relative dimension m.

Definition 16.1. Let T be any scheme of finite type over k and $m \geq 0$ an integer. The presheaf $z_{equi}(T, m)$ on Sm/k is defined as follows. For each smooth S, $z_{equi}(T, m)(S)$ is the free abelian group generated by the closed and irreducible subvarieties Z of $S \times T$ which are dominant and equidimensional of relative dimension m over a component of S. If $S' \to S$ is any map, the pullback of equidimensional cycles (see 1A.5) induces the required natural map $z_{equi}(T, m)(S) \to z_{equi}(T, m)(S')$.

It is not hard to see that $z_{equi}(T, m)$ is a Zariski sheaf, and even an étale sheaf, for each T and m. One can also check that each $z_{equi}(T, m)$ is contravariant for flat maps in T, and covariant for proper maps in T, both with the appropriate change in the dimension index m, (see [RelCy, 3.6.2 and 3.6.4]); see [Blo86, 1.3]. In particular, if $T' \hookrightarrow T$ is a closed immersion, there are canonical inclusions $z_{equi}(T', m) \hookrightarrow z_{equi}(T, m)$ for all m.

Example 16.2. The case $m = 0$ is of particular interest, since $z_{equi}(T, 0)(U)$ is free abelian on the irreducible $Z \subset U \times T$ which are quasi-finite and dominant over U. Hence $\mathbb{Z}_{tr}(T)(U) \subseteq z_{equi}(T,0)(U)$, because $\mathbb{Z}_{tr}(T)(U)$ is the free abelian group of cycles in $U \times T$ which are finite and surjective over U. In fact, $\mathbb{Z}_{tr}(T)$ is a sub-sheaf of $z_{equi}(T,0)$ because the structure morphisms associated to $V \to U$ are compatible: $\mathbb{Z}_{tr}(T)(U) \to \mathbb{Z}_{tr}(T)(V)$ is also the pullback of cycles (see 1A.9).

If T is projective, then $\mathbb{Z}_{tr}(T) = z_{equi}(T,0)$. Indeed, each closed subvariety $Z \subset U \times T$ is proper over U, so Z is quasi-finite over U if and only if Z is finite over U (see [Har77, Ex. III.11.2]).

We now define transfer maps for $z_{equi}(T,m)$ which are compatible with the transfers in $\mathbb{Z}_{tr}(T)$ when $m=0$. Given an elementary correspondence V from X to Y and a cycle $\mathcal{Z}$ in $z_{equi}(T,m)(Y)$, the pullback $\mathcal{Z}_V$ is a well-defined cycle of $V \times T$ by 1A.5 and 1A.8. We define $\phi_V(\mathcal{Z}) \in z_{equi}(T,m)(X)$ to be the push-forward of $\mathcal{Z}_V$ along the finite map $V \times T \to X \times T$. This gives a homomorphism $\phi_V : z_{equi}(T,m)(Y) \to z_{equi}(T,m)(X)$.

If $m=0$, the restriction of ϕ_V to $\mathbb{Z}_{tr}(T)$ is the transfer map constructed in 1.1 and 1A.9, as we see from 1A.11.

We leave the verification of the following to the reader; cf. [BivCy, 5.7].

Exercise 16.3. If T is a smooth scheme, show that ϕ makes each $z_{equi}(T,m)$ into a presheaf with transfers. If $U \to T$ is flat, show that the pull-back $z_{equi}(T,m) \to z_{equi}(U,m)$ is a morphism of presheaves with transfer.

Example 16.4. For each X, there is a natural map $z_{equi}(\mathbb{A}^i,0)(X) \to CH^i(\mathbb{A}^i \times X) \cong CH^i(X)$, sending a subvariety Z of $\mathbb{A}^i \times X$, quasi-finite over X, to its cycle. Comparing the transfer map for $z_{equi}(\mathbb{A}^i,0)$ to the transfer map for $CH^i(X)$ defined in 2.5, we see that $z_{equi}(\mathbb{A}^i,0) \to CH^i(-)$ is a morphism of presheaves with transfers.

We define the **Suslin-Friedlander motivic complexes** $\mathbb{Z}^{SF}(i)$ by:

$$\mathbb{Z}^{SF}(i) = C_* z_{equi}(\mathbb{A}^i,0)[-2i].$$

We regard $\mathbb{Z}^{SF}(i)$ as a bounded above cochain complex, whose top term is $z_{equi}(\mathbb{A}^i,0)$ in cohomological degree $2i$. As in 3.1, $C_*(F)$ stands for the chain complex of presheaves associated to the simplicial presheaf $U \mapsto F(U \times \Delta^\bullet)$.

Example 16.5. It follows from 16.2 and 16.3 that there is a morphism of presheaves with transfers from $\mathbb{Z}_{tr}(\mathbb{P}^i) = z_{equi}(\mathbb{P}^i,0)$ to $z_{equi}(\mathbb{A}^i,0)$, with kernel $\mathbb{Z}_{tr}(\mathbb{P}^{i-1})$. Applying C_* gives an exact sequence of complexes of presheaves with transfers $0 \to C_*\mathbb{Z}_{tr}(\mathbb{P}^{i-1}) \to C_*\mathbb{Z}_{tr}(\mathbb{P}^i) \to \mathbb{Z}^{SF}(i)[2i]$.

Exercise 16.6. Let E be the function field of a smooth variety over k. Show that the stalk at $\operatorname{Spec} E$ of the sheaf $z_{equi}(\mathbb{A}^i_k, 0)$ on Sm/k is equal to the global sections $z_{equi}(\mathbb{A}^i_E, 0)(\operatorname{Spec} E)$ of the sheaf $z_{equi}(\mathbb{A}^i_E, 0)$ on Sm/E. Similarly, show that the stalk of $C_m z_{equi}(\mathbb{A}^i_k, 0)$ at $\operatorname{Spec} E$ equals $C_m z_{equi}(\mathbb{A}^i_E, 0)(\operatorname{Spec} E)$.

Conclude that the stalk of $\mathbb{Z}^{SF}(i)$ at $\operatorname{Spec} E$ equals $\mathbb{Z}^{SF}(i)(\operatorname{Spec} E)$, and is independent of the choice of k.

Here are the two main results in this lecture. Figure 16.1 gives the scheme of the proof of 16.7. It shows how this result ultimately depends on theorem 13.11, whose proof will be completed in lectures 20-23 below.

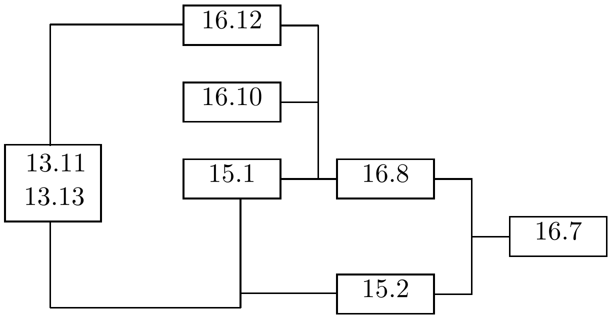

Figure 16.1: Scheme of the proof of 16.7

Theorem 16.7. *There is a quasi-isomorphism in the Zariski topology:*

$$\mathbb{Z}(i) \simeq \mathbb{Z}^{SF}(i).$$

In particular, $H^{n,i}(X, \mathbb{Z}) \cong \mathbb{H}^n(X, \mathbb{Z}^{SF}(i))$ *for all* n *and* i.

Proof. As $\mathbb{P}^i$ is proper, $z_{equi}(\mathbb{P}^i, 0) = \mathbb{Z}_{tr}(\mathbb{P}^i)$ by 16.2. Hence 16.7 follows from combining 15.2 and 16.8. □

Theorem 16.8. *There is a quasi-isomorphism in the Zariski topology:*

$$C_*\left[z_{equi}(\mathbb{P}^i, 0)/z_{equi}(\mathbb{P}^{i-1}, 0)\right] \xrightarrow{\simeq} C_* z_{equi}(\mathbb{A}^i, 0).$$

We now prepare for the proof of 16.8.

Let $F_i(U)$ denote the (free abelian) subgroup of $z_{equi}(\mathbb{A}^i, 0)(U)$ generated by the cycles in $U \times \mathbb{A}^i$ which do not touch $U \times 0$. By inspection, the transfers $z_{equi}(\mathbb{A}^i, 0)(U) \to z_{equi}(\mathbb{A}^i, 0)(V)$ send $F_i(U)$ to $F_i(V)$. Hence F_i is a sub-presheaf with transfers of $z_{equi}(\mathbb{A}^i, 0)$.

Lemma 16.9. *There is a commutative diagram with exact rows in* $\mathbf{PST}(k)$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_{tr}(\mathbb{P}^i - 0)/\mathbb{Z}_{tr}(\mathbb{P}^{i-1}) \arrow[r] \arrow[d, hook] & \mathbb{Z}_{tr}(\mathbb{P}^i)/\mathbb{Z}_{tr}(\mathbb{P}^{i-1}) \arrow[r] \arrow[d, hook, "\varphi"] & \text{coker}_1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & F_i \arrow[r] & z_{equi}(\mathbb{A}^i, 0) \arrow[r] & \text{coker}_2 \arrow[r] & 0.
\end{tikzcd}$$

All three vertical maps are injections.

Proof. By example 16.2, there is a natural map from $\mathbb{Z}_{tr}(\mathbb{P}^i) = z_{equi}(\mathbb{P}^i, 0)$ to $z_{equi}(\mathbb{A}^i, 0)$ with kernel $\mathbb{Z}_{tr}(\mathbb{P}^{i-1})$. Thus φ is an injection; by exercise 16.3, φ is a morphism of presheaves with transfers. Now the inclusion $\mathbb{Z}_{tr}(\mathbb{P}^i - 0) \subset \mathbb{Z}_{tr}(\mathbb{P}^i)$ is a morphism in **PST** by the Yoneda lemma; see 2.7. Since $\mathbb{Z}_{tr}(\mathbb{P}^i - 0)(U)$ consists of cycles $Z \subset U \times (\mathbb{P}^i - 0)$ finite over U, their restriction belongs to the subgroup $F_i(U)$, i.e., φ sends $\mathbb{Z}_{tr}(\mathbb{P}^i - 0)$ to F_i. Hence the diagram commutes.

By inspection, $\text{coker}_1(X)$ is free abelian on the elementary correspondences $Z \subset X \times \mathbb{P}^i$ which touch $X \times 0$ and $\text{coker}_2(X)$ is free abelian on the equidimensional $W \subset X \times \mathbb{A}^i$ which touch $X \times 0$. Since $Z \mapsto \varphi(Z)$ is a monomorphism on these generators, it follows that $\text{coker}_1(X) \to \text{coker}_2(X)$ is an injection for all X. $\square$

Lemma 16.10. $C_*(F_i)$ *is chain contractible as a complex of presheaves.*

Proof. Recall that $F_i(X)$ is a subgroup of the group of cycles on $X \times \mathbb{A}^i$. Let $h_X : F_i(X) \to F_i(X \times \mathbb{A}^1)$ be the pullback of cycles along $\mu : X \times \mathbb{A}^i \times \mathbb{A}^1 \to X \times \mathbb{A}^i$ defined by $(x, r, t) \to (x, r \cdot t)$. This is a good pullback because the map μ is flat over $X \times (\mathbb{A}^i - \{0\})$. By construction, the following diagram commutes.

$$\begin{tikzcd}
X \times \mathbb{A}^i \arrow[r, "t=1"] \arrow[dr, "id"'] & X \times \mathbb{A}^i \times \mathbb{A}^1 \arrow[d, "\mu"'] & X \times \mathbb{A}^i \arrow[l, "t=0"'] \arrow[dl, "id \times 0"] \\
& X \times \mathbb{A}^i &
\end{tikzcd}$$

It follows that $F_i(t = 1)h_X$ is the identity and $F_i(t = 0)h_X = 0$. Thus the requirements of lemma 2.21 are satisfied for F_i. $\square$

Lemma 16.11. *For every Hensel local scheme S, the map* $\operatorname{coker}_1(S) \to \operatorname{coker}_2(S)$ *in diagram 16.9 is an isomorphism.*

Proof. Since $\operatorname{coker}_1(S) \to \operatorname{coker}_2(S)$ is injective by 16.9, it suffices to prove that it is surjective. Let Z be a equidimensional correspondence from S to $\mathbb{A}^i$. As Z is quasi-finite over a Hensel scheme, the projection decomposes Z into the disjoint union of Z_0 (which doesn't contain any point over the closed point of the Hensel scheme) and Z_1 (which is finite over the base). We claim that the Z_0 part comes from F_i. Take Z_0 and consider its irreducible components. The intersection $Z_0 \cap \{0\}$ must be empty, otherwise it would project to the closed point of the base. Hence Z_0 is zero in the cokernel. But now Z_1 comes from $\mathbb{Z}_{tr}(\mathbb{P}^i)/\mathbb{Z}_{tr}(\mathbb{P}^{i-1})$. □

Lemma 16.12. *The map* $C_*(\operatorname{coker}_1) \to C_*(\operatorname{coker}_2)$ *is a quasi-isomorphism of complexes of Zariski sheaves.*

Proof. Let φ' be the map between the cokernels in 16.9. By 16.11, φ' is an isomorphism on all Hensel local schemes. By 13.13, φ' induces quasi-isomorphisms $C_* \operatorname{coker}_1(X) \simeq C_* \operatorname{coker}_2(X)$ for all local X. □

Proof of 16.8. Applying C_* to the diagram in 16.9 yields a commutative diagram of chain complexes with exact rows. The left two complexes are acyclic by 15.1 and 16.10. The right two complexes are quasi-isomorphic by 16.12. Theorem 16.8 now follows from the 5-lemma. □

Lecture 17

Higher Chow groups

During the first part of this series of lectures we defined motivic cohomology and we studied its basic properties. We also established relations with some classic objects of algebraic geometry, such as Milnor K-Theory, 5.1, and étale cohomology, 10.2.

The goal of the next few lectures is to find a relation between motivic cohomology and the classical Chow groups CH^i, generalizing the isomorphism $H^{2,1}(X, \mathbb{Z}) \cong \mathrm{Pic}(X) = CH^1(X)$ of 4.2. That is, we will prove that:

$$H^{2i,i}(X, \mathbb{Z}) \cong CH^i(X)$$

for any smooth variety X. There are at least three ways to prove this. The original approach, which needs resolution of singularities, was developed in the book *"Cycles, Transfers and Motivic Homology Theories"* [VSF00]. A second recent approach is to use the Cancellation Theorem of [Voe02] and the Gersten resolution 23.11 for motivic cohomology sheaves.

A third approach, which is the one we shall develop here, uses Bloch's higher Chow groups $CH^i(X, m)$ to establish the more general isomorphism $H^{n,i}(X, \mathbb{Z}) \cong CH^i(X, 2i - n)$. This approach uses the equidimensional cycle groups of the previous lecture, but does not use resolution of singularities.

The main goal of this lecture is to prove that the higher Chow groups are presheaves with transfers. (See theorem 17.20.) In particular, they are functorial for maps between smooth schemes. (We will give a second proof of this in 19.15.)

We begin with Bloch's definition of higher Chow groups (see [Blo86]).

Definition 17.1. Let X be an equidimensional scheme. We write $z^i(X, m)$ for the free abelian group generated by all codimension i subvarieties on $X \times \Delta^m$ which intersect all faces $X \times \Delta^j$ properly for all $j < m$ (in the sense of 17A.1).

Each face $X \times \Delta^j$ is defined by a regular sequence, and intersection of cycles defines a map $z^i(X, m) \to z^i(X, j)$ (see 17A.1, or [Ful84, Example 7.1.2]). We write $z^i(X, \bullet)$ for the resulting simplicial abelian group $m \mapsto z^i(X, m)$. We write $z^i(X, *)$ for the chain complex associated to $z^i(X, \bullet)$.

The **higher Chow groups** of X are defined to be the groups:

$$CH^i(X, m) = \pi_m(z^i(X, \bullet)) = H_m(z^i(X, *)).$$

If X is any scheme, it is easy to check that $CH^i(X, 0)$ is the classical Chow group $CH^i(X)$ (see 17.3). Indeed, $z^i(X, 0)$ is the group of all codimension i cycles on X while $z^i(X, 1)$ is generated by those codimension i subvarieties Z on $X \times \mathbb{A}^1$ which intersect both $X \times \{0\}$ and $X \times \{1\}$ properly. Moreover the maps $\partial_j : z^i(X, 1) \rightrightarrows z^i(X, 0)$ send Z to $Z \cap (X \times \{j\})$.

Example 17.2. If $i \leq d = \dim X$, then $z_{equi}(X, d - i)(\Delta^m) \subseteq z^i(X, m)$ for every m, because cycles in $X \times \Delta^m$ which are equidimensional over Δ^m must meet every face properly. By 1A.12, the inclusion is compatible with the face maps, which are defined in 16.1 and 17.1, so this yields an inclusion of simplicial groups, $z_{equi}(X, d - i)(\Delta^\bullet) \subseteq z^i(X, \bullet)$.

Exercise 17.3. (a) If $d = \dim X$, show that every irreducible cycle in $z^d(X, 1)$ is either disjoint from $X \times \{0, 1\}$ or else is quasi-finite over $\mathbb{A}^1$. Use this to describe $z^d(X, 1) \to z^d(X, 0)$ explicitly and show that $CH^d(X, 0) \cong CH^d(X)$. (The group $CH^d(X)$ is defined in [Ful84, 1.6].)

(b) Show that $C_*\mathbb{Z}_{tr}(X)(\operatorname{Spec} k)$ is a subcomplex of $z^d(X, *)$. On homology, this yields maps $H_m^{sing}(X/k) \to CH^d(X, m)$. For $m = 0$, show that this is the surjection $H_0^{sing}(X/k) \to CH_0(X) = CH^d(X)$ of 2.20, which is an isomorphism when X is projective. By 7.3, it is not an isomorphism when X is $\mathbb{A}^1$ or $\mathbb{A}^1 - 0$.

The push-forward of cycles makes the higher Chow groups covariant for finite morphisms (see 17A.9). It also makes them covariant for proper morphisms (with the appropriate change in codimension index i; see [Blo86, 1.3]).

At the chain level, it is easy to prove that the complexes $z^i(-,*)$, and hence Bloch's higher Chow groups, are functorial for flat morphisms. However, the complexes $z^i(-,*)$ are not functorial for all maps. We will see in 17.20 below that the higher Chow groups are functorial for maps between smooth schemes.

We will need the following non-trivial properties of higher Chow groups:

1. Homotopy Invariance: The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism
$$CH^i(X,m) \xrightarrow{\cong} CH^i(X \times \mathbb{A}^1, m)$$
for any scheme X over k. The proof is given in [Blo86, 2.1].

2. Localization Theorem: For any $U \subset X$ open, the cokernel of $z^i(X,\bullet) \to z^i(U,\bullet)$ is acyclic. This is proven by Bloch in [Blo94]. (Cf. [Blo86, 3.3].)

   If the complement $Z = X - U$ has pure codimension c, it is easy to see that we have an exact sequence of simplicial abelian groups (and also of complexes of abelian groups):
$$0 \to z^{i-c}(Z,\bullet) \to z^i(X,\bullet) \to z^i(U,\bullet) \to \text{coker} \to 0.$$
   Thus the localization theorem yields long exact sequences of higher Chow groups. The fact that we need to use Bloch's Localization Theorem is unfortunate, because its proof is very hard and complex.

Transfers maps associated to correspondences are not defined on all of $z^i(Y,*)$. We need to restrict to a subcomplex on which $\mathcal{W}^*$ may be defined.

Definition 17.4. Let $\mathcal{W}$ be a finite correspondence from X to Y. Write $z^i(Y,m)_{\mathcal{W}}$ for the subgroup of $z^i(Y,m)$ generated by the irreducible subvarieties $T \subset Y \times \Delta^m$ such that the pullback $X \times T$ intersects $\mathcal{W} \times \Delta^j$ properly in $X \times Y \times \Delta^m$ for every face $\Delta^j \hookrightarrow \Delta^m$. By construction, $z^i(Y,*)_{\mathcal{W}}$ is a subcomplex of $z^i(Y,*)$.

The proof of the following proposition, which is a refinement of the results in [Lev98], is due to Marc Levine.

Proposition 17.5. *Let $\mathcal{W}$ be a finite correspondence from X to Y, with Y affine. Then the inclusion $z^i(Y,*)_{\mathcal{W}} \subset z^i(Y,*)$ is a quasi-isomorphism.*

Proof. (Levine) Let $w : W \to Y$ be a morphism of schemes with Y smooth, and W locally equidimensional but not necessarily smooth. Write $z^i(Y,m)_w$ for the subgroup of $z^i(Y,m)$ generated by the irreducible subvarieties $T \subset Y \times \Delta^m$ for which every component of $w^{-1}(T)$ has codimension at least i in $W \times \Delta^m$ and intersects every face properly. By construction, $z^i(Y,*)_w$ is a subcomplex of $z^i(Y,*)$.

For example, if W is the support of a finite correspondence $\mathcal{W}$, let $w : W \to Y$ be the natural map. Then W is locally equidimensional, and the group $z^i(Y,m)_w$ is the same as the group $z^i(Y,m)_{\mathcal{W}}$ of 17.4.

Levine proves that $z^i(Y,m)_w \hookrightarrow z^i(Y,m)$ is a quasi-isomorphism on p.102 of [Lev98] (in I.II.3.5.14), except that W is required to also be smooth in order to cite lemma I.II.3.5.2 of *op. cit.*. In *loc. cit.*, a finite set $\{C_j\}$ of locally closed irreducible subsets of Y and a sequence of integers $m_j \leq i$ is constructed, with the property that T is in $z^i(Y,m)_w$ if and only if T is in $z^i(Y,m)$ and the intersections of T with $C_j \times \Delta^p$ have codimension at least m_j for all j and for every face Δ^p of Δ^m. A reading of the proof of lemma I.II.3.5.2 shows that in fact W need only be locally equidimensional. $\square$

Definition 17.6. Let $\mathcal{W}$ be a finite correspondence between two smooth schemes X and Y. For each cycle $\mathcal{Y}$ in $z^i(Y,m)_{\mathcal{W}}$, we define the cycle $\mathcal{W}^*(\mathcal{Y})$ on $X \times \Delta^m$ to be:

$$\mathcal{W}^*(\mathcal{Y}) = \pi_*((\mathcal{W} \times \Delta^m) \cdot (X \times \mathcal{Y})).$$

Here $\pi : X \times Y \times \Delta^m \to X \times \Delta^m$ is the canonical projection.

For each $\mathcal{W}$, it is clear that $\mathcal{W}^*$ defines a homomorphism from the group $z^i(Y,m)_{\mathcal{W}}$ to the group of all cycles on $X \times \Delta^n$.

Example 17.7. Let $f : X \to Y$ be a morphism of smooth varieties, and let Γ_f be its graph. For $\mathcal{Y}$ in $z^i(Y,0)_{\Gamma_f}$, $\Gamma_f^*(\mathcal{Y})$ is just the classical pullback of cycles $f^*(\mathcal{Y})$ defined in [Ser65, V-28] (see 17A.3).

Remark 17.8. The map $\mathcal{W}^*$ of 17.6 is compatible with the map $\mathcal{W}^*$ defined in 17A.7 in the following sense. Given $\mathcal{W}$ in $Cor(X,Y)$, $\mathcal{W} \times \mathrm{diag}(\Delta^m)$ is a finite correspondence from $X \times \Delta^m$ to $Y \times \Delta^m$. If $\mathcal{Y}$ is a cycle in $z^i(Y,m)_{\mathcal{W}}$, we may regard it as a cycle in $Y \times \Delta^m$. The projection formula 17A.10 says that the following diagram commutes:

$$\begin{tikzcd} z^i(Y,m)_{\mathcal{W}} \arrow[r, hook] \arrow[d, "\mathcal{W}^*"'] & z^i(Y \times \Delta^m)_{\mathcal{W}} \arrow[d, "(\mathcal{W} \times \mathrm{diag}(\Delta^m))^*"] \\ z^i(X,m) \arrow[r, hook] & z^i(X \times \Delta^m). \end{tikzcd}$$

Lemma 17.9. *The maps $\mathcal{W}^*$ define a chain map $z^i(Y,*)_{\mathcal{W}} \to z^i(X,*)$.*

Proof. Let $\partial_j : \Delta^m \hookrightarrow \Delta^{m+1}$ be a face, and consider the following diagram, whose vertical compositions are $\mathcal{W}^*$:

$$\begin{tikzcd}
z(Y \times \Delta^{m+1}) \arrow[r, "\partial_j^*"] \arrow[d, "f^*"'] & z(Y \times \Delta^m) \arrow[d, "f^*"] \\
z(X \times Y \times \Delta^{m+1}) \arrow[r, "\partial_j^*"] \arrow[d, "\mathcal{W} \times \Delta^{m+1} \cdot -"'] & z(X \times Y \times \Delta^m) \arrow[d, "\mathcal{W} \times \Delta^m \cdot -"] \\
z(X \times Y \times \Delta^{m+1}) \arrow[r, "\partial_j^*"] \arrow[d, "\pi_*"'] & z(X \times Y \times \Delta^m) \arrow[d, "\pi_*"] \\
z(X \times \Delta^{m+1}) \arrow[r, "\partial_j^*"] & z(X \times \Delta^m).
\end{tikzcd}$$

The horizontal maps ∂_j^* are only defined for cycles meeting the face properly (see 17A.4) and the intersection products in the middle are only defined on cycles in good position for $\mathcal{W}$. The top square commutes because of the functoriality of Bloch's complex for flat maps, and the bottom square commutes by 17A.9.

Suppose that $\mathcal{Z}$ is a cycle in $X \times Y \times \Delta^{m+1}$ which intersects the face $X \times Y \times \Delta^m$ as well as $\mathcal{W} \times \Delta^{m+1}$ and $\mathcal{W} \times \Delta^m$ properly. By 17A.2:

$$\mathcal{W} \times \Delta^m \cdot ((X \times Y \times \Delta^m) \cdot \mathcal{Z}) = X \times Y \times \Delta^m \cdot ((\mathcal{W} \times \Delta^{m+1}) \cdot \mathcal{Z}).$$

That is, the middle square commutes for $\mathcal{Z}$. Finally, if $\mathcal{Y} \in z^i(Y, m+1)_w$, the cycle $(\mathcal{W} \times \Delta^{m+1}) \cdot f^*\mathcal{Y}$ is finite over $X \times \Delta^{m+1}$, so π_* may be applied to it. A diagram chase now shows that $\mathcal{W}^*$ is a morphism of chain complexes. □

Corollary 17.10. *If Y is affine, any finite correspondence $\mathcal{W}$ from X to Y induces maps $\mathcal{W}^* : CH^i(Y,m) \to CH^i(X,m)$ for all m.*

Proof. On homology, 17.5 and $\mathcal{W}^*$ give: $CH^i(Y,m) \cong H_m(z^i(Y,*)_{\mathcal{W}}) \to H_m(z^i(X,*)) = CH^i(X,m)$. □

Example 17.11. If $f : X \to Y$ is a morphism of smooth varieties, and Y is affine, we will write f^* for the map Γ_f^* from $z^i(Y,m)_{\Gamma_f}$ to $z^i(X,m)$, and also for the induced map from $CH^i(Y,m)$ to $CH^i(X,m)$. It agrees with Levine's map f^* (see pp. 67 and 102 of [Lev98]). This is not surprising, since we are using lemma 17.5, which is taken from p. 102 of [Lev98]. The map f^* may also be obtained from [Blo86, 4.1] using [Blo94].

If f is flat, then f^* is just the flat pullback of cycles. That is, if $\mathcal{Y} = [V]$ then $f^*(\mathcal{Y})$ is the cycle associated to the scheme $f^{-1}(V)$. This fact is a special case of 17A.4.

We can now show that the higher Chow groups are functors on the subcategory of affine schemes in Cor_k.

Lemma 17.12. *Let X, Y and Z be smooth affine schemes. Given finite correspondences $\mathcal{W}_1$ in $Cor(X,Y)$ and $\mathcal{W}_2$ in $Cor(Y,Z)$, then*

$$(\mathcal{W}_2 \circ \mathcal{W}_1)^* = \mathcal{W}_1^* \mathcal{W}_2^* : CH^i(Z,m) \to CH^i(X,m).$$

In particular, if $f_1 : X \to Y$ and $f_2 : Y \to Z$ are morphisms, then $(f_2 \circ f_1)^* = f_1^* f_2^*$.

Proof. By 17.5 and 17.11, it suffices to show that $(\mathcal{W}_2 \circ \mathcal{W}_1)^* = \mathcal{W}_1^* \mathcal{W}_2^*$ as maps from $z^i(Z,m)_{\mathcal{W}} \to z^i(X,m)$, where $\mathcal{W} \in Cor(Y \coprod X, Z)$ is the coproduct of $\mathcal{W}_2$ and $\mathcal{W}_2 \circ \mathcal{W}_1$. An element of $z^i(Z,m)_{\mathcal{W}}$ is a cycle in $z^i(Z,m)$ which is in good position with respect to both $\mathcal{W}_2$ and $\mathcal{W}_2 \circ \mathcal{W}_1$. Hence the result follows from theorem 17A.13, given the reinterpretation in 17.8. $\square$

We now extend the definition of the transfer map $\mathcal{W}^*$ from affine varieties to all smooth varieties using Jouanolou's device [Jou73, 1.5] and [Wei89, 4.4]: over every smooth variety X there is a vector bundle torsor $X' \to X$ with X' affine.

Lemma 17.13. *Let X be a variety and $p : X' \to X$ a vector bundle torsor. Then $p^* : CH^*(X,*) \to CH^*(X',*)$ is an isomorphism.*

Proof. By definition, there is a dense open U in X so that $p^{-1}(U) \cong U \times \mathbb{A}^n$. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & z^*(X' - p^{-1}(U)) \arrow[r] & z^*(X',*) \arrow[r] & z^*(p^{-1}(U)) \\
0 \arrow[r] & z^*(X - U) \arrow[r] \arrow[u] & z^*(X,*) \arrow[r] \arrow[u] & z^*(U). \arrow[u]
\end{tikzcd}$$

By homotopy invariance of the higher Chow groups (see p. 149), the right vertical arrow is a quasi-isomorphism. By Noetherian induction, the result is true for $X - U$, i.e., the left vertical arrow is a quasi-isomorphism. By the Localization Theorem and and the five lemma, $p^* : CH^*(X, *) \to CH^*(X', *)$ is an isomorphism. $\square$

Lemma 17.14. *Let $p : Y' \to Y$ be a vector bundle torsor and let X be affine.*

- *Every morphism $f : X \to Y$ has a lift $f' : X \to Y'$ such that $pf' = f$.*
- *Every finite correspondence has a lift, i.e., $p_* : Cor(X, Y') \to Cor(X, Y)$ is surjective.*

Proof. Clearly, $X \times_Y Y' \to X$ is a vector bundle torsor. But X is affine and therefore every vector bundle torsor over X is a vector bundle (see [Wei89, 4.2]). Define $f' : X \to Y'$ to be the composition of the zero-section of $X \times_Y Y'$ followed by the projection. Clearly, $pf' = f$.

Now suppose that $W \subset X \times Y$ is an elementary correspondence. Since W is finite over X, it is affine. By the first part of this proof, the projection $p : W \to Y$ lifts to a map $p' : W \to Y'$. Together with the projection $W \to X$, p' induces a lift $i : W \to X \times Y'$ of $W \subset X \times Y$. Then $i(W)$ is an elementary correspondence from X to Y' whose image under p_* is W. $\square$

Lemma 17.15. *Let X and Y be two smooth varieties over k and let $p : X' \to X$ and $q : Y' \to Y$ be vector bundle torsors with X' and Y' affine. Then for every finite correspondence $\mathcal{W}$ from X to Y, there exists a correspondence $\mathcal{W}'$ from X' to Y' so that $q \circ \mathcal{W}' = \mathcal{W} \circ p$ in $Cor_k(X', Y)$.*

$$\begin{tikzcd} X' \arrow[r, "\mathcal{W}'"] \arrow[d, "p"'] & Y' \arrow[d, "q"] \\ X \arrow[r, "\mathcal{W}"] & Y \end{tikzcd}$$

Proof. Since $Cor(X', Y') \to Cor(X', Y)$ is onto by 17.14, $\mathcal{W} \circ p$ has a lift $\mathcal{W}'$. $\square$

Definition 17.16. Let X and Y be two smooth varieties over k and let $\mathcal{W}$ be a finite correspondence from X to Y. We define $\mathcal{W}^* : CH^i(Y, m) \to CH^i(X, m)$ as follows.

By Jouanolou's device [Jou73, 1.5], there exist vector bundle torsors $p : X' \to X$ and $q : Y' \to Y$ where X' and Y' are affine. Both X' and Y' are smooth, because X and Y are. By lemma 17.13, p^* and q^* are isomorphisms.

By 17.15 there is a finite correspondence $\mathcal{W}'$ from X' to Y' so that $q \circ \mathcal{W}' = \mathcal{W} \circ p$ in $Cor_k(X', Y)$. Since Y' is affine, the map $\mathcal{W}'^* : CH^*(Y', m) \to CH^*(X', m)$ was defined in 17.10. We set $\mathcal{W}^* = (p^*)^{-1}\mathcal{W}'^* q^* : CH^*(Y, m) \to CH^*(X, m)$.

$$\begin{tikzcd} CH^*(X',*) & CH^*(Y',*) \arrow[l, "\mathcal{W}'^*"'] \\ CH^*(X,*) \arrow[u, "p^*", "\cong"'] & CH^*(Y,*) \arrow[l, "\mathcal{W}^*"'] \arrow[u, "q^*", "\cong"'] \end{tikzcd}$$

If $f : X \to Y$ is a morphism, we define $f^* : CH^i(Y, m) \to CH^i(X, m)$ to be Γ_f^*, that is, $f^* = (p^*)^{-1}(f')^* q^*$, where $f' : X' \to Y'$ lies over f.

Lemma 17.17. *If X and Y are affine, the map defined in 17.16 agrees with the map $\mathcal{W}^*$ defined in 17.10.*

Proof. By 17.12, the map defined in 17.16 equals:

$$(p^*)^{-1}\mathcal{W}'^* q^* = (p^*)^{-1}(q \circ \mathcal{W}')^* = (p^*)^{-1}(\mathcal{W} \circ p)^* = (p^*)^{-1}p^*\mathcal{W}^* = \mathcal{W}^*. \quad \square$$

Lemma 17.18. *The definition of $\mathcal{W}^*$ in 17.16 is independent of the choices.*

Proof. Suppose given affine torsors $X'' \to X$ and $Y'' \to Y$ and a lift $\mathcal{W}'' \in Cor(X'', Y'')$ of $\mathcal{W}$. We have to show that $\mathcal{W}'$ and $\mathcal{W}''$ induce the same map.

We begin by making two reductions. First, we may assume that $X' = X''$ and $Y' = Y''$, by passing to $X' \times_X X''$ and $Y' \times_Y Y''$ and choosing lifts of $\mathcal{W}'$ and $\mathcal{W}''$. (This reduction uses 17.17.)

We may also assume that X is affine and that $X' = X$, by replacing X by X'. Thus we need to show that for any two lifts $\mathcal{W}_0$ and $\mathcal{W}_1$ of $\mathcal{W}$, ${\mathcal{W}_0}^* q^* = {\mathcal{W}_1}^* q^*$.

By lemma 17.19, there is a finite correspondence $\widetilde{\mathcal{W}}$ so that the following diagram commutes:

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[rr, "\widetilde{\mathcal{W}}"] & & Y' \arrow[d, "q"] \\ X \arrow[u, shift left, "s_1"'] \arrow[u, shift right, "s_0"] \arrow[urr, shift left, "\mathcal{W}_0"] \arrow[urr, shift right, "\mathcal{W}_1"'] \arrow[rr, "\mathcal{W}"] & & Y. \end{tikzcd}$$

Since s_0 and s_1 are both inverses to the projection $p : X \times \mathbb{A}^1 \to X$, we have $s_0^* p^* = s_1^* p^*$ by 17.12. Since higher Chow groups are homotopy invariant, p^* is an isomorphism and we get $s_0^* = s_1^*$. Since X and Y' are affine, and $\mathcal{W}_i = \widetilde{\mathcal{W}} \circ s_i$, we may apply 17.12 again to get

$$\mathcal{W}_0^* = s_0^* \widetilde{\mathcal{W}}^* = s_1^* \widetilde{\mathcal{W}}^* = \mathcal{W}_1^*. \qquad \square$$

Recall from 2.24 that two correspondences $\mathcal{W}_0$ and $\mathcal{W}_1$ from X to Y are said to be $\mathbb{A}^1$-homotopic, written $\mathcal{W}_0 \simeq \mathcal{W}_1$, if they are the restrictions of an element of $Cor(X \times \mathbb{A}^1, Y)$ along $X \times 0$ and $X \times 1$.

Lemma 17.19. *Let $\mathcal{W}$ be a finite correspondence between a smooth affine scheme X and a smooth Y. If $q : Y' \to Y$ is a vector bundle torsor, then any two lifts $\mathcal{W}_0$ and $\mathcal{W}_1$ are $\mathbb{A}^1$-homotopic.*

Proof. Let V be the image of the union of the supports of $\mathcal{W}_0$ and $\mathcal{W}_1$ in $X \times Y$, and let V' denote the fiber product of V and Y' over Y; $p : V' \to V$ is a vector bundle torsor. Since X is affine and the induced map $V \to X$ is finite, V is affine too. Hence $p : V' \to V$ is a vector bundle. Fix a section $s : V \to V'$.

$$\begin{tikzcd} V' \arrow[r] \arrow[d, shift left, "p"] & X \times Y' \arrow[r] \arrow[d] & Y' \arrow[d, "q"] \\ V \arrow[r, hook] \arrow[u, shift left, "s"] & X \times Y \arrow[r] & Y \end{tikzcd}$$

Clearly, p is an $\mathbb{A}^1$-homotopy equivalence (in the sense of 2.24) with inverse s, that is, sp is $\mathbb{A}^1$-homotopic to the identity.

Both $\mathcal{W}_0$ and $\mathcal{W}_1$ induce correspondences $\widetilde{\mathcal{W}_0}$ and $\widetilde{\mathcal{W}_1}$ from X to V'. Now the composition $g \circ (\widetilde{\mathcal{W}_i} \times \mathbb{A}^1) \in Cor(X \times \mathbb{A}^1, V')$ is an $\mathbb{A}^1$-homotopy from $sp\widetilde{\mathcal{W}_i}$ to $\widetilde{\mathcal{W}_i}$, for $i = 0, 1$. Since $p\widetilde{\mathcal{W}_0} = p\widetilde{\mathcal{W}_1}$, we have

$$\widetilde{\mathcal{W}_0} \simeq sp\widetilde{\mathcal{W}_0} = sp\widetilde{\mathcal{W}_1} \simeq \widetilde{\mathcal{W}_1}.$$

Since $\mathcal{W}_i$ is the composition of $\widetilde{\mathcal{W}_i}$ with the map $V' \to Y$, $\mathcal{W}_0$ is $\mathbb{A}^1$-homotopic to $\mathcal{W}_1$. $\square$

At last, we have the tools to show that the higher Chow groups are presheaves with transfers, i.e., functors on the category Cor_k of smooth separated schemes over k.

Theorem 17.20. *The maps $\mathcal{W}^*$ defined in 17.16 give the higher Chow groups $CH^i(-,m)$ the structure of presheaves with transfers.*

That is, for any two finite correspondences $\mathcal{W}_1$ and $\mathcal{W}_2$ from X to Y and from Y to Z, respectively, and for all $\alpha \in CH^i(Z,m)$:

$$\mathcal{W}_1{}^*(\mathcal{W}_2{}^*(\alpha)) = (\mathcal{W}_2 \circ \mathcal{W}_1)^*(\alpha).$$

In particular, if $f_1 : X \to Y$ and $f_2 : Y \to Z$ are morphisms, then $(f_2 \circ f_1)^* = f_1^* f_2^*$.

Proof. By 17.15, there is a commutative diagram in Cor_k of the form

$$\begin{tikzcd} X' \arrow[r, "\mathcal{W}_1'"] \arrow[d, "p"'] & Y' \arrow[r, "\mathcal{W}_2'"] \arrow[d, "q"'] & Z' \arrow[d, "r"'] \\ X \arrow[r, "\mathcal{W}_1"] & Y \arrow[r, "\mathcal{W}_2"] & Z \end{tikzcd}$$

where the vertical maps are affine vector bundle torsors. By 17.12, we have ${\mathcal{W}_1'}^*{\mathcal{W}_2'}^* = (\mathcal{W}_2' \circ \mathcal{W}_1')^*$. Since the definitions of $\mathcal{W}_i^*$ and $(\mathcal{W}_2 \circ \mathcal{W}_1)^*$ are independent of the choices by 17.18, the statement now follows from an unwinding of 17.16:

$$\mathcal{W}_1^*\mathcal{W}_2^* = (p^*)^{-1}{\mathcal{W}_1'}^*q^*(q^*)^{-1}{\mathcal{W}_2'}^*r^* = (p^*)^{-1}(\mathcal{W}_2' \circ \mathcal{W}_1')^*r^* = (\mathcal{W}_2 \circ \mathcal{W}_1)^*. \quad \square$$

Appendix 17A- Cycle maps

If $\mathcal{W}$ is a finite correspondence from X to Y, we can define a map $\mathcal{W}^*$ from "good" cycles on Y to cycles on X. The formula is to pull the cycle back to $X \times Y$, intersect it with $\mathcal{W}$, and push forward to X. In this appendix, we will make this precise, in 17A.7. First we must explain what makes a cycle "good".

Definition 17A.1. Two subvarieties Z_1 and Z_2 of X are said to **intersect properly** if every component of $Z_1 \cap Z_2$ has codimension $\operatorname{codim} Z_1 + \operatorname{codim} Z_2$ in X. This is vacuously true if $Z_1 \cap Z_2 = \emptyset$.

If the ambient variety X is regular, the intersection cycle $Z_1 \cdot Z_2$ is defined to be the sum $\sum n_j[W_j]$, where the indexing is over the irreducible components W_j of $Z_1 \cap Z_2$, and the n_j are their (local) intersection multiplicities. Following Serre [Ser65], the multiplicity n_j is defined as follows. If A is the local ring of X at the generic point of W_j, and I_l are the ideals of A defining Z_l, then

$$n_j = \sum_i (-1)^i \text{length}\,\mathrm{Tor}\,_i^A(A/I_1, A/I_2).$$

If X is not regular, the multiplicity will only make sense when only finitely many Tor-terms are non-zero.

We say that two equidimensional cycles $\mathcal{V} = \sum m_i V_i$ and $\mathcal{W} = \sum n_j W_j$ intersect properly if each V_i and W_j intersect properly. In this case, the intersection cycle $\mathcal{V} \cdot \mathcal{W}$ is defined to be $\sum m_j n_j (V_i \cdot W_j)$.

Exercise 17A.2. Let $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ be three cycles on a smooth scheme X. Show that $(\mathcal{V}_1 \cdot \mathcal{V}_2) \cdot \mathcal{V}_3 = \mathcal{V}_1 \cdot (\mathcal{V}_2 \cdot \mathcal{V}_3)$ whenever both sides are defined. (This is proven in [Ser65, V-24].)

Definition 17A.3. Suppose that $f : X \to Y$ is a morphism with X and Y regular, and that $\mathcal{Y}$ is a codimension i cycle on Y. We say that $f^*(\mathcal{Y})$ is

defined if each component of $f^{-1}(Supp(\mathcal{Y}))$ has codimension $\geq i$ in X. As in [Ser65, V-28], we define the cycle $f^*(\mathcal{Y})$ to be $\Gamma_f \cdot (X \times \mathcal{Y})$ (see 17A.1), identifying the graph Γ_f with X.

As noted in [Ser65, V-29], the intersection cycle makes sense even if X is not regular, since the multiplicities may be computed over Y by flat base change for Tor (see [Wei94, 3.2.9]).

Example 17A.4. If f is flat and $\mathcal{Y} = [V]$, then $f^*(\mathcal{Y})$ is the cycle associated to the scheme $f^{-1}(V)$. If X is a subvariety of Y, then the cycle $f^*(\mathcal{Y})$ on X is the same as the cycle $X \cdot \mathcal{Y}$ considered as a cycle on X. If $X \hookrightarrow Y$ is a regular embedding, the coefficients of $f^*(\mathcal{Y})$ agree with the intersection multiplicities defined in [Ful84, 7.1.2].

Definition 17A.5. Let $f : Y' \to Y$ be a morphism of smooth varieties and $\mathcal{W}$ a cycle on Y'. We say that a cycle $\mathcal{Y}$ on Y is in **good position** for $\mathcal{W}$ (relative to f) if the cycle $f^*(\mathcal{Y})$ is defined, and intersects $\mathcal{W}$ properly on Y'. If $\mathcal{Y}$ is in good position for $\mathcal{W}$, the intersection product $\mathcal{W} \cdot f^*\mathcal{Y}$ is defined (see 17A.1). If the map f is flat, the cycle $f^*(\mathcal{Y})$ is always defined.

Let W be an irreducible subvariety of Y' and let w be the composition $W \to Y' \to Y$. By 17A.1 and 17A.3, a codimension i cycle $\mathcal{Y}$ is in good position for W if and only if $\operatorname{codim}_W w^{-1}(Supp(\mathcal{Y})) \geq i$, that is, if $w^*(\mathcal{Y})$ is defined.

As a special case, we will say that a cycle $\mathcal{Y}$ is in good position for a finite correspondence $\mathcal{W}$ from X to Y if $\mathcal{Y}$ is in good position for the cycle underlying $\mathcal{W}$, relative to the projection $X \times Y \to Y$.

Remark 17A.6. Let $f : X \to Y$ be a morphism of smooth varieties and let $\mathcal{Z}$ be a cycle on X, supported on a closed subscheme Z so that the composition $Z \to X \to Y$ is a proper map. It is clear that $f_*(\mathcal{Z})$ is well-defined even though f is not proper.

Definition 17A.7. Let $\mathcal{W}$ be a finite correspondence between two smooth schemes X and Y. For every cycle $\mathcal{Y}$ on Y in good position for $\mathcal{W}$, we define

$$\mathcal{W}^*(\mathcal{Y}) = \pi_*(\mathcal{W} \cdot f^*\mathcal{Y}),$$

where $f : X \times Y \to Y$ and $\pi : X \times Y \to X$ are the canonical projections. The intersection and the push-forward are well-defined by 17A.5 and 17A.6. The map $\mathcal{W}^*$ induces the transfer map for Chow groups, see 17.10 and 17.16.

For any smooth T, $\mathcal{W} \times T$ is a finite correspondence from $X \times T$ to $Y \times T$ over T. By abuse of notation, we shall also write $\mathcal{W}^*$ for $(\mathcal{W} \times T)^*$.

Example 17A.8. We can now reinterpret the composition of correspondences. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite correspondences from X to Y and from Y to Z, respectively, we have:

$$\mathcal{W}_2 \circ \mathcal{W}_1 = (\mathcal{W}_1 \times Z)^*(\mathcal{W}_2) = (X \times \mathcal{W}_2)^*(\mathcal{W}_1).$$

Here are two formulas which are useful in the study of $\mathcal{W}^*$.

Lemma 17A.9. *Consider the following diagram of varieties*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

where the square is fiber and both X and Y are smooth. Let $\mathcal{X}$ be a cycle on X whose support is finite over Y and for which $(g')^*\mathcal{X}$ is defined. Then $g^* f_* \mathcal{X}$ is defined and $g^* f_* \mathcal{X} = f'_*(g')^*\mathcal{X}$.

Proof. If V is a component of $\mathcal{X}$, then the map $f : V \to f(V)$ is finite. Hence $f' : (g')^{-1}(V) \to g^{-1}(f(V))$ is finite too, so that $\operatorname{codim}(g')^{-1}(V) = \operatorname{codim} g^{-1}(f(V))$. By hypothesis, $\operatorname{codim}(g')^{-1}(V) \geq i$, which proves that $g^* f_* \mathcal{X}$ is defined. The equality now follows from [Ful75, 2.2(4)]. $\square$

Lemma 17A.10 (Projection Formula). *Let $f : X \to Y$ be a morphism of smooth schemes. Suppose given a cycle $\mathcal{X}$ on X, whose support is finite over Y, and a cycle $\mathcal{Y}$ on Y which is in good position for $\mathcal{X}$ (see 17A.5). Then $f_*\mathcal{X}$ and $\mathcal{Y}$ intersect properly, and the projection formula holds:*

$$f_*(\mathcal{X} \cdot f^*\mathcal{Y}) = f_*\mathcal{X} \cdot \mathcal{Y}.$$

Proof. Since the restriction of f to the support of $\mathcal{X}$ is finite, it is clear that $f_*(\mathcal{X})$ and $\mathcal{Y}$ intersect properly too. The result is now a consequence of the basic identity 2.2(2) of [Ful75], or [Ser65, V-30]. $\square$

Exercise 17A.11. Let i be the inclusion of a closed subvariety W in a smooth scheme X and let $f : X \to Y$ be a map of smooth schemes. Prove that if $\mathcal{Y}$ is a cycle on Y so that both $f^*\mathcal{Y}$ and $(fi)^*(\mathcal{Y})$ are defined, then $i_*(fi)^*(\mathcal{Y}) = W \cdot f^*\mathcal{Y}$. *Hint*: Use [Ser65, V-30] or [Ful75, 2.2(2)].

Recall from 1A.8 that if $V \to Y$ is a morphism with Y regular, then the pull-back $\mathcal{Z}_V$ of a relative cycle $\mathcal{Z}$ in $T \times Y$ is a well defined cycle on $T \times V$ with integer coefficients.

Lemma 17A.12. *Let T and Y be regular and let $\mathcal{Z}$ be a cycle in $T \times Y$ which is dominant equidimensional over Y. If $f : V \to Y$ is a morphism, then the pull-back $\mathcal{Z}_V$ agrees with the pull-back cycle $(f \times T)^*(\mathcal{Z})$.*

Proof. Note that $\mathcal{Z}$ is a relative cycle by 1A.5, so that $\mathcal{Z}_V$ is defined. Its coefficients are characterized by the equalities $(\mathcal{Z}_V)_v = \mathcal{Z}_{f(v)}$ for every $v \in V$. By [RelCy, 3.5.8 and 3.5.9], the coefficients of $\mathcal{Z}_V$ are the same as the multiplicities in 17A.1, i.e., the coefficients of $(f \times T)^*(\mathcal{Z})$ given by 17A.3. $\square$

Theorem 17A.13. *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two finite correspondences from X to Y and from Y to Z, respectively. Suppose that $\mathcal{Z}$ is a cycle on Z which is in good position with respect to both $\mathcal{W}_2$ and $\mathcal{W}_2 \circ \mathcal{W}_1$. Then*

$$(\mathcal{W}_2 \circ \mathcal{W}_1)^*(\mathcal{Z}) = \mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z})).$$

The term $\mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z}))$ makes sense by the following lemma.

Lemma 17A.14. *Let $\mathcal{Z}$ be in good position for $\mathcal{W}_2$ and $\mathcal{W}_2 \circ \mathcal{W}_1$. Then $\mathcal{W}_2^*(\mathcal{Z})$ is in good position with respect to $\mathcal{W}_1$.*

Proof. We may assume that the correspondences are elementary, i.e., $\mathcal{W}_1$ and $\mathcal{W}_2$ are subvarieties W_1 and W_2 of $X \times Y$, and $Y \times Z$, respectively. In this spirit, we will write $W_2 \circ W_1$ for the subvariety of $X \times Z$ which is the support of the composition of correspondences, $\mathcal{W}_2 \circ \mathcal{W}_1$. Consider the following diagram:

$$\begin{tikzcd}
W_2 \circ W_1 \arrow[drr, "c"] & & \\
W_1 \times_Y W_2 \arrow[u, "u"] \arrow[r, "e"] \arrow[d, "q"'] & W_2 \arrow[r, "d"'] \arrow[d, "p"] & Z \\
W_1 \arrow[r, "b"] & Y &
\end{tikzcd}$$

By hypothesis, $\operatorname{codim} d^{-1}(\mathcal{Z}) \geq \operatorname{codim} \mathcal{Z}$ and $\operatorname{codim} c^{-1}(\mathcal{Z}) \geq \operatorname{codim} \mathcal{Z}$.

We claim that $\operatorname{codim} b^{-1}pd^{-1}\mathcal{Z} \geq \operatorname{codim} \mathcal{Z}$. Since the central square is cartesian, $b^{-1}p = qe^{-1}$. Since q is finite, this yields

$$\operatorname{codim} b^{-1}pd^{-1}\mathcal{Z} = \operatorname{codim} qe^{-1}d^{-1}\mathcal{Z} = \operatorname{codim} e^{-1}d^{-1}\mathcal{Z}.$$

But $e^{-1}d^{-1} = u^{-1}c^{-1}$, and u is finite, so:

$$\operatorname{codim} e^{-1}d^{-1}\mathcal{Z} = \operatorname{codim} u^{-1}c^{-1}\mathcal{Z} = \operatorname{codim} c^{-1}\mathcal{Z}.$$

But $\operatorname{codim} c^{-1}\mathcal{Z} \geq \operatorname{codim} \mathcal{Z}$ by hypothesis, as claimed. $\square$

Proof of 17A.13. The right side is defined by 17A.14. We will follow the notation established in figure 17A.1, where we have omitted the factor Δ^n in every entry to simplify notation. Note that the central square is cartesian.

$$\begin{tikzcd}
X \times Z \arrow[drrr, "c"] \arrow[dddr, "v"'] & & & \\
& X \times Y \times Z \arrow[ul, "u"'] \arrow[r, "e"] \arrow[d, "q"'] & Y \times Z \arrow[r, "d"'] \arrow[d, "p"] & Z \\
& X \times Y \arrow[r, "b"] \arrow[d, "r"] & Y & \\
& X & &
\end{tikzcd}$$

Figure 17A.1: Composition of correspondences

By definition 17.6, we have

$$\mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z})) = r_*(\mathcal{W}_1 \cdot b^*(p_*(\mathcal{W}_2 \cdot d^*\mathcal{Z}))).$$

Since the central square is cartesian, we have $b^*p_* = q_*e^*$ by 17A.9. Since the pull-back e^* is a ring homomorphism, we have

$$b^*(p_*(\mathcal{W}_2 \cdot d^*\mathcal{Z})) = q_*(e^*(\mathcal{W}_2 \cdot d^*\mathcal{Z})) = q_*\left(e^*(\mathcal{W}_2) \cdot e^*d^*\mathcal{Z}\right).$$

Consider the two cycles $\mathcal{X} = e^*(\mathcal{W}_2) \cdot (de)^*(\mathcal{Z})$ and $\mathcal{Y} = \mathcal{W}_1$ and the function q. The intersection $\mathcal{X} \cdot q^*\mathcal{Y} = e^*(\mathcal{W}_2) \cdot (de)^*(\mathcal{Z}) \cdot q^*(W_1)$ is proper because $\mathcal{Z}$ is in good position with respect to $W_2 \circ W_1$. Therefore the conditions for 17A.10 are satisfied, and the projection formula yields $\mathcal{Y} \cdot q_*\mathcal{X} = q_*(q^*\mathcal{Y} \cdot \mathcal{X})$, i.e.,

$$\mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z})) = r_*q_*\left(q^*(\mathcal{W}_1) \cdot (e^*(\mathcal{W}_2) \cdot e^*d^*\mathcal{Z})\right).$$

Since the push-forward and pullback are functorial, we have $r_*q_* = v_*u_*$ and $e^*d^* = u^*c^*$. Our cycle then becomes

$$v_*u_*\left(q^*(\mathcal{W}_1) \cdot e^*(\mathcal{W}_2) \cdot u^*c^*\mathcal{Z}\right).$$

We may use the projection formula (17A.10) once again, this time for u^*, with $\mathcal{X} = q^*(\mathcal{W}_1) \cdot e^*(\mathcal{W}_2)$ and $\mathcal{Y} = c^*\mathcal{Z}$ (the conditions are satisfied by the same argument we used above). This yields $u_*(\mathcal{X} \cdot u^*\mathcal{Y}) = (u_*\mathcal{X}) \cdot \mathcal{Y}$, i.e.,

$$\mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z})) = v_*\left(u_*(q^*(\mathcal{W}_1) \cdot e^*(\mathcal{W}_2)) \cdot c^*\mathcal{Z}\right).$$

Since the composition of $\mathcal{W}_1$ and $\mathcal{W}_2$ as correspondences is exactly $u_*(q^*(\mathcal{W}_1) \cdot e^*(\mathcal{W}_2))$, the last equation becomes

$$\mathcal{W}_1^*(\mathcal{W}_2^*(\mathcal{Z})) = v_*((\mathcal{W}_2 \circ \mathcal{W}_1) \cdot c^*\mathcal{Z}) = (\mathcal{W}_2 \circ \mathcal{W}_1)^*(\mathcal{Z}).$$

This concludes the proof of 17A.13. $\square$

Lecture 18

Higher Chow groups and equidimensional cycles

The next step in the proof of theorem 19.1 (that motivic cohomology and higher Chow groups agree) is the reduction to equidimensional cycles. The main references for this lecture are [HigCh] and [FS00].

Definition 18.1. For an equidimensional X, and $i \leq \dim X$, we write $z^i_{equi}(X, m)$ for $z_{equi}(X, \dim X - i)(\Delta^m)$, the free abelian group generated by all codimension i subvarieties on $X \times \Delta^m$ which are dominant and equidimensional over Δ^m (of relative dimension $\dim X - i$). We write $z^i_{equi}(X, \bullet)$ and $z^i_{equi}(X, *)$ for the simplicial abelian group $m \mapsto z^i_{equi}(X, m)$ and its associated chain complex, respectively.

By 17.2, $z^i_{equi}(X, m)$ is a subgroup of $z^i(X, m)$ and $z^i_{equi}(X, \bullet)$ is a simplicial subgroup of $z^i(X, \bullet)$.

Example 18.2. The inclusion $z^i_{equi}(X, *) \subset z^i(X, *)$ will not be a quasi-isomorphism in general. Indeed, if $i > d$ then $z^i_{equi}(X, m) = 0$ while $z^i(X, m)$ is not generally zero. For example, consider $X = \operatorname{Spec} k$. If $i > 0$ we have $z^i_{equi}(\operatorname{Spec} k, *) = 0$. In contrast, $z^i(\operatorname{Spec} k, i)$ is the group of points on Δ^i which do not lie on any proper face. We will see in 19.7 that $H_i z^i(\operatorname{Spec} k, *) = H^{i,i}(\operatorname{Spec} k) = K^M_i(k)$.

Theorem 18.3. *(Suslin [HigCh, 2.1]) Let X be an equidimensional affine scheme of finite type over k, then the inclusion map:*

$$z^i_{equi}(X, *) \hookrightarrow z^i(X, *)$$

is a quasi-isomorphism for $i \leq \dim X$.

Corollary 18.4. *Let X be an affine variety, then for all $i \geq 0$*

$$CH^i(X, m) = H_m(z^i_{equi}(X \times \mathbb{A}^i, *)).$$

In particular, $CH^i(\operatorname{Spec} k, m) = H_m(z^i_{equi}(\mathbb{A}^i, *))$.

Proof. This is an immediate corollary of 18.3, definition 17.1 and the homotopy invariance of the higher Chow groups; see page 149. □

We need lemmas 18.7, 18.13 and 18.13 to prove theorem 18.3. All of their proofs rely on a technical theorem 18A.1, which will be proven in the appendix.

We begin by introducing some auxiliary notions. Let X be a scheme over S.

Definition 18.5. An **N−skeletal map** φ over X, relative to $X \to S$, is a collection $\{\varphi_n : X \times \Delta^n \to X \times \Delta^n\}_{n=0}^N$ of S-morphisms, such that φ_0 is the identity 1_X and for every face map $\partial_j : \Delta^{n-1} \to \Delta^n$ with $n \leq N$ the following diagram commutes.

$$\begin{tikzcd}
X \times \Delta^{n-1} \arrow[r, "\varphi_{n-1}"] \arrow[d, "1_X \times \partial_j"'] & X \times \Delta^{n-1} \arrow[d, "1_X \times \partial_j"] \\
X \times \Delta^n \arrow[r, "\varphi_n"] & X \times \Delta^n
\end{tikzcd}$$

Note that φ_N determines φ_n for all $n < N$. When $S = X$, we shall just call φ an n-skeletal map over X.

The condition that an $(N-1)$-skeletal map over X can be extended to an N-skeletal map is a form of the homotopy extension property, and follows from the Chinese Remainder Theorem when X is affine.

For example, a 1-skeletal map over $X = \operatorname{Spec} R$ (relative to $S = X$) is determined by a polynomial $f \in R[t]$ such that $f(0) = 0$ and $f(1) = 1$; φ_1 is Spec of the R-algebra map $R[t] \to R[t]$ sending t to f.

Definition 18.6. Given an N-skeletal map φ over X and $n \leq N$, we define $\varphi z^i(X, n)$ to be the subgroup of $z^i(X, n)$ generated by all V in $X \times \Delta^n$ such that $\varphi_n^*(V)$ is defined and is in $z^i(X, n)$. If $n > N$ we set $\varphi z^i(X, n) = 0$. In other words, $\varphi z^i(X, n)$ is the group of cycles in $X \times \mathbb{A}^n$ which intersect all the faces properly and whose pullbacks along φ_n intersect all the faces properly.

By definition 18.5 we know that the face map $\partial_j : z^i(X,n) \to z^i(X,n-1)$ sends $\varphi z^i(X,n)$ to $\varphi z^i(X,n-1)$. Thus $\varphi z^i(X,*)$ is a chain subcomplex of $z^i(X,*)$. Moreover it follows from 18.5 that the φ_n^* assemble to define a chain map $\varphi^* : \varphi z^i(X,*) \to z^i(X,*)$.

Similarly, we can define $\varphi z^i_{equi}(X,n)$ to be the subgroup of $z^i_{equi}(X,n)$ generated by all V such that $\varphi_n^*(V)$ is defined and is in $z^i_{equi}(X,n)$. The same argument shows that $\varphi z^i_{equi}(X,*)$ is a subcomplex of $z^i_{equi}(X,n)$ and that the φ_n form a chain map $\varphi^* : \varphi z^i_{equi}(X,*) \to z^i_{equi}(X,*)$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \varphi z^i(X,1) \arrow[r] \arrow[d, hook, shift right, "i"'] \arrow[d, shift left, "\varphi_1^*"] & \varphi z^i(X,0) \arrow[r] \arrow[d, "="] & 0 \\
z^i(X,2) \arrow[r] & z^i(X,1) \arrow[r] & z^i(X,0) \arrow[r] & 0.
\end{tikzcd}$$

Figure 18.1: A 1-skeletal map φ and its chain map φ^*.

Lemma 18.7. *(See [HigCh, 2.8]) Let C_* be a finitely generated subcomplex in $z^i(X,*)$ with $i \le \dim X$. Choose N so that $C_n = 0$ for $n > N$. Then there is an N-skeletal map φ over X such that $C_* \subseteq \varphi z^i(X,*)$, and the chain map $\varphi^* : \varphi z^i(X,*) \to z^i(X,*)$ satisfies*

$$\varphi^* C_* \subseteq z^i_{equi}(X,*).$$

Example 18.8. If $N = 1$, and $\alpha \in k - \{0,1\}$, the subvariety $V = X \times \{\alpha\}$ of $X \times \mathbb{A}^1$ is in $z^1(X,1)$ but not $z^1_{equi}(X,1)$. If $X = \operatorname{Spec} R$, fix $r \in R$ and let $\varphi_1 : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be the 1-skeletal map defined by the R-algebra map $R[t] \to R[t]$ sending t to $f(t) = t + r(t^2 - t)$. The condition that $\varphi_1^*(V)$ is in $z^1_{equi}(X,1)$, i.e., dominant and equidimensional over Δ^1, is equivalent to the condition that the map $r : X \to \mathbb{A}^1$ is equidimensional, i.e., that $r - \beta$ be nonzero in the domain R for all $\beta \in k$. Indeed, the fibre of $\varphi_1^{-1}(V)$ over $t \neq 0, 1$ is supported on $R/(r - (\alpha - t/t^2 - t))$, and is empty if $t = 0, 1$. Such r always exists when $\dim X \ge 1$.

Proof of 18.7. Suppose that C_n is generated by $\{V_n^k\} \subseteq z^i(X,n)$. Set $d = \dim X - i$ and note that $d \geq 0$ since $i \leq \dim X$. Then $V_n = \cup V_n^k$ is closed in $X \times \Delta^n$ of dimension $n+d$.

We proceed by induction on N. Since N is finite, we may assume that the $\partial_j(V_n^k)$ are supported in V_{n-1}. Inductively, we may suppose that we have constructed an $(N-1)$-skeletal map $\{\varphi_n\}$ such that the fibers of the projections $\varphi_n^{-1}(V_n) \to \Delta^n$ have dimension $\leq d$. Let $\partial\Delta^N$ be the union of the faces Δ^N. The compatibility granted by definition 18.5 implies that these maps fit together to form a map from $X \times \partial\Delta^N$ to itself such that the fibers of $\varphi^{-1}(X \times \partial\Delta^N) \cap V_N \to \partial\Delta^N$ have dimension $\leq d$. By Generic Equidimensionality 18A.1, this map extends to a N-skeletal map $\varphi_N : X \times \Delta^N \to X \times \Delta^N$ over X such that the fibers of $\varphi_N^{-1}(V_N) \to \mathbb{A}^N$ have dimension $\leq d$. Because each component W of $\varphi^{-1}(V_n^k)$ satisfies the inequality $\dim W \leq n+d = \dim V_n^k$, each cycle $\varphi_n^*(V_n^k)$ is defined and lies in $z^i_{equi}(X,n)$. Since C_n is generated by the V_n^k, it lies in $\varphi z^i(X,n)$ and satisfies $\varphi^*(C_n) \subset z^i_{equi}(X,n)$. $\square$

Definition 18.9. Let φ^0 and φ^1 be N-skeletal maps over X. An **N−skeletal homotopy** Φ between φ^0 and φ^1 is an N-skeletal map $\{\Phi_n : X \times \Delta^n \times \mathbb{A}^1 \to X \times \Delta^n \times \mathbb{A}^1\}_{n=0}^N$ over $X \times \mathbb{A}^1$ relative to the projection $X \times \mathbb{A}^1 \to X$, which is compatible with the φ^j in the sense that the following diagram commutes for every n.

$$\begin{tikzcd}
X \times \Delta^n \arrow[r, "i_0"] \arrow[d, "\varphi_n^0"'] & X \times \Delta^n \times \mathbb{A}^1 \arrow[d, "\Phi_n"] & X \times \Delta^n \arrow[l, "i_1"'] \arrow[d, "\varphi_n^1"] \\
X \times \Delta^n \arrow[r, "i_0"] & X \times \Delta^n \times \mathbb{A}^1 & X \times \Delta^n \arrow[l, "i_1"']
\end{tikzcd}$$

The subgroup $\Phi z^i(X,n)$ of $z^i(X,n)$ is defined to be the subgroup generated by all V in $X \times \Delta^n$ such that $(\varphi^0)^*(V)$, $(\varphi^1)^*(V)$ and $\Phi^*(V \times \mathbb{A}^1)$ are all defined. As in definition 18.6, $\Phi z^i(X,*)$ is a subcomplex of $z^i(X,*)$. In fact, Φz^i is in $(\varphi^0 z^i) \cap (\varphi^1 z^i)$.

Lemma 18.10. *If Φ is an N-skeletal homotopy between φ^0 and φ^1, then the maps $(\varphi^0)^*$ and $(\varphi^1)^*$ from $\Phi z^i(X,*)$ to $z^i(X,*)$ are chain homotopic.*

Proof. For $0 \leq j \leq n$, let h_j denote the composite

$$X \times \Delta^{n+1} \xrightarrow{1_X \times \theta_j} X \times \Delta^n \times \mathbb{A}^1 \xrightarrow{\Phi_n} X \times \Delta^n \times \mathbb{A}^1 \xrightarrow{pr} X \times \Delta^n,$$

where the isomorphisms $\theta_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ were defined in 2.16, and pr is the projection. That is, for V in $\Phi z^i(X, n)$ we define

$$h_j^*[V] = (1_X \times \theta_j)^* \Phi_n^*[V \times \mathbb{A}^1] \in z^i(X, n+1).$$

The h_j^* form a simplicial homotopy (see [Wei94, 8.3.11]) from $\partial_0 h_0 = \varphi^1$ to $\partial_{n+1} h_n = \varphi^0$. Hence their alternating sum $h = \sum(-1)^j h_j^*$ satisfies $h\partial + \partial h = (\varphi^1)^* - (\varphi^0)^*$. (This is illustrated in figure 18.2 when $N = 2$.) $\square$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & \Phi z^i(X,2) \arrow[r, "\partial"] \arrow[dl, "h"] \arrow[d, shift right, "\varphi_2^1"'] \arrow[d, shift left, "\varphi_2^0"] & \Phi z^i(X,1) \arrow[r, "\partial"] \arrow[dl, "h"] \arrow[d, shift right, "\varphi_1^1"'] \arrow[d, shift left, "\varphi_1^0"] & \Phi z^i(X,0) \arrow[dl, "0"] \arrow[d, shift right, "\varphi_0^1"'] \arrow[d, shift left, "\varphi_0^0 = 1_X"] \\
z^i(X,3) \arrow[r, "\partial"] & z^i(X,2) \arrow[r, "\partial"] & z^i(X,1) \arrow[r, "\partial"] & z^i(X,0)
\end{tikzcd}$$

Figure 18.2: The chain homotopy between i and φ^* when $N = 2$.

Proposition 18.11. *Let φ be an N-skeletal map, and $\{V_n^k\}$ a finite set of varieties in $\varphi z^i(X, n)$, $n \leq N$. Then there exists an N-skeletal homotopy Φ between φ and the identity map, such that each $\Phi^*(V_n^k \times \mathbb{A}^1)$ is defined and lies in $z^i_{equi}(X \times \mathbb{A}^1, n)$.*

Proof. Set $d = \dim(X) - i$. As in the proof of 18.7, we construct Φ_n by induction on n. Inductively, we are given an $(N-1)$-skeletal map Φ_n such that the fibers of the projections $\Phi^{-1}(V_n^k \times \mathbb{A}^1) \to \Delta^n \times \mathbb{A}^1$ have dimension $\leq d$. Let $\partial(\Delta^N \times \mathbb{A}^1)$ denote the union of $(\partial\Delta^N) \times \mathbb{A}^1$ and $\Delta^N \times \{0, 1\}$. The compatibility with the faces of Δ^N and with i_0, i_1 granted by definition 18.9 implies that the Φ_n and φ_N fit together to form a map $\partial\Phi_N$ from $X \times \partial(\Delta^N \times \mathbb{A}^1)$ to itself such that the fibers of

$$\partial\Phi_N^{-1}(X \times \partial(\Delta^N \times \mathbb{A}^1) \cap V_N^k \times \mathbb{A}^1) \to \Delta^N \times \mathbb{A}^1$$

have dimension $\leq d$. By Generic Equidimensionality 18A.1, with $\mathbb{A}^n = \Delta^N \times \mathbb{A}^1$, this map extends to a map Φ_N from $X \times \Delta^N \times \mathbb{A}^1$ to itself which extends $\partial\Phi_N$ (i.e., an N-skeletal homotopy from the identity to φ over X), such that the fibers of $\Phi_N^{-1}(V_N^k \times \mathbb{A}^1) \to \Delta^N \times \mathbb{A}^1$ have dimension $\leq d$ over all points of $\Delta^N \times \mathbb{A}^1$ not on $\partial(\Delta^N \times \mathbb{A}^1)$. This completes the inductive step, and shows that each cycle $\Phi_n^*(V_n^k \times \mathbb{A}^1)$ is defined and lies in $z^i_{equi}(X \times \mathbb{A}^1, n)$. $\square$

In order to simplify the proof of theorem 18.3 we need to introduce the "topological" notion of weak homotopy.

Definition 18.12. Two maps $f, g : K \to L$ of complexes of abelian groups are called **weakly homotopic** if for every finitely generated subcomplex C of K, the restrictions $f|_C$ and $g|_C$ are chain homotopic.

It is easy to check that weakly homotopic maps induce the same maps on homology. If K and L are bounded complexes of free abelian groups, this notion is equivalent to the usual notion of chain homotopy between maps. To see that this notion is weaker than chain homotopy, consider a pure subgroup A of B which is not a summand, such as $\oplus_1^\infty \mathbb{Z} \subset \prod_1^\infty \mathbb{Z}$. Then the canonical map from $(A \to B)$ to $(A \to 0)$ is weakly homotopic to zero but not chain contractible.

Lemma 18.13. *(See [HigCh, 2.3 and 2.6]) Let φ be an N-skeletal map over X. Then the maps i and φ^* are are weakly homotopic on φz^i:*

$$\varphi z^i(X, *) \underset{\varphi^*}{\overset{i}{\rightrightarrows}} z^i(X, *),$$

and also on φz^i_{equi}:

$$\varphi z^i_{equi}(X, *) \underset{\varphi^*}{\overset{i}{\rightrightarrows}} z^i_{equi}(X, *).$$

Note that the following diagram commutes

$$\begin{tikzcd}
\varphi z^i_{equi}(X, -) \arrow[r, hook] \arrow[d, shift right, hook, "i"'] \arrow[d, shift left, "\varphi^*"] & \varphi z^i(X, -) \arrow[d, shift right, hook, "i"'] \arrow[d, shift left, "\varphi^*"] \\
z^i_{equi}(X, -) \arrow[r, hook] & z^i(X, -).
\end{tikzcd}$$

Moreover if $a \in \varphi z^i(X, n) \cap z^i_{equi}(X, n)$, and $\varphi^* a \in z^i_{equi}(X, n)$, then $a \in \varphi z^i_{equi}(X, n)$.

Proof of 18.3. We have to prove that the induced map on homology classes is an isomorphism:

$$H_n(z^i_{equi}(X,*)) \to H_n(z^i(X,*)) \tag{18.13.1}$$

First we prove surjectivity. Let $a \in z^i(X,n)$ be such that $d(a) = 0$. Lemma 18.7 provides an integer N and an N-skeletal map $\{\varphi_n\}$ such that $a \in \varphi z^i(X,n)$ and $\varphi^*(a) \in z^i_{equi}(X,n)$. By 18.13, $a - \varphi^* a$ is a boundary in $z^i(X,n)$, i.e., a and $\varphi^*(a)$ represent the same class in homology. Hence the map 18.13.1 is surjective.

For injectivity we need to consider $a \in z^i_{equi}(X,n)$ so that $d(a) = 0$ and $b \in z^i(X,n+1)$ with $d(b) = a$. Apply lemma 18.7 to b and a. We find an $(n+1)$-skeletal map φ such that $a, b \in \varphi z^i(X,*)$ and $\varphi^* a, \varphi^* b \in z^i_{equi}(X,*)$. But now we have:

$$\varphi^* a = \varphi^*(db) = d(\varphi^* b) = 0.$$

From lemma 18.13, a and $\varphi^* a = 0$ represent the same class in the homology of $z^i_{equi}(X,*)$. Therefore a is a boundary in $z^i_{equi}(X,*)$. Hence the map (18.13.1) is also injective. □

Now we shall prove lemma 18.13 using 18A.1.

Proof of 18.13. Consider a subcomplex $C_* \hookrightarrow \varphi z^i(X,*)$ generated by some closed irreducible subvarieties V^k_n so that $\partial_j(V^k_n)$ is a linear combination of generators. It suffices to prove that the inclusion of C_* into $z^i(X,*)$ is homotopic to φ^*. But this is just an application of 18.11 and 18.10.

Lemma 18.14. *(See [HigCh] 2.4) For any V and φ_n as before and any $\Delta^m \to \Delta^n \times \mathbb{A}^1$, $(id_X \times \theta)^{-1}(V \times \mathbb{A}^1)$ has dimension $\leq m + t$.*

So $\theta^*(V)$ is well-defined and we can verify that h is an homotopy. □

Appendix 18A- Generic Equidimensionality

This appendix is devoted to a proof of the following Generic Equidimensionality Theorem, due to Suslin. (See [HigCh] 1.1.)

Theorem 18A.1. *Let S be an affine scheme of finite type over a field. Let V be a closed subscheme of $S \times \mathbb{A}^n$, Z an effective divisor of $\mathbb{A}^n$ and $\varphi : S \times Z \to S \times \mathbb{A}^n$ any morphism over S. For every $t \geq 0$ so that $\dim V \leq n + t$, there exists a map $\Phi : S \times \mathbb{A}^n \to S \times \mathbb{A}^n$ over S so that:*

1. *$\Phi|_{S \times Z} = \varphi$;*
2. *the fibers of the projection $\Phi^{-1}(V) \to \mathbb{A}^n$ have dimension $\leq t$ over the points of $\mathbb{A}^n - Z$.*

The S-morphism $\varphi : S \times Z \to S \times \mathbb{A}^n$ is determined by its component $\varphi' : S \times Z \to \mathbb{A}^n$. If $S \subset \mathbb{A}^m$, we can extend φ' to a morphism $\psi' : \mathbb{A}^m \times Z \to \mathbb{A}^n$. If we knew the theorem for $\mathbb{A}^m$, there would exist an extension $\Psi' : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n$ of ψ' such that, setting $\Psi(X, Y) = (X, \Psi'(X, Y))$, the fibers of $\Psi^{-1}(V) \to \mathbb{A}^n$ over points of $\mathbb{A}^n - Z$ have dimension $\leq t$, and the restriction Φ of Ψ to $S \times \mathbb{A}^n$ would satisfy the conclusion of the theorem. Thus we may suppose that $S = \mathbb{A}^m$.

Write $\mathbb{A}^m = \operatorname{Spec} k[x_1, \dots, x_m]$ and $\mathbb{A}^n = \operatorname{Spec} k[y_1, \dots, y_n]$. If the divisor Z is defined by a polynomial $h \in k[Y]$ then the component $\varphi' : \mathbb{A}^m \times Z \to \mathbb{A}^n$ of φ extends to $f = (f_1, \dots, f_n) : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n$ for polynomials $f_i \in k[X, Y]$ defined up to a multiple of h. For each n-tuple $F = (F_1, \dots, F_n)$ of homogeneous forms in $k[X]$ of degree N, consider the maps

$$\Phi_F : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n$$

$$\Phi_F(X, Y) = (f_1(X, Y) + h(Y)F_1(X), \dots, f_n(X, Y) + h(Y)F_n(X)).$$

By construction, the restriction of Φ_F to $Z \times S$ is φ', i.e., property (1) holds. It suffices to show that if $N >> 0$ and the F_i are in general position then $\Phi(X,Y) = (X, \Phi_F(X,Y))$ has the desired property (2).

If $I = (g_1, \dots, g_s)$ is the ideal of $k[X,Y]$ defining V, then the ideal J of $k[X,Y]$ defining $\Phi^{-1}(V)$ is generated by the polynomials

$$g_j(X, \Phi_F) = g_j(x_1, \dots, x_m, \Phi_1, \Phi_2, \dots, \Phi_n), \quad \Phi_i = f_i(X,Y) + h(Y)F_i(X).$$

If b is a k-point of $\mathbb{A}^n$, the ideal J_b of $k[X]$ defining the fiber over b is generated by the $g_j(X, \Phi_F(X,b))$. We need to show that if $b \notin Z$, then J_b has height $\geq m - t$.

Example 18A.2. Suppose that $m = 1$ and $t = 0$. We may assume that $\dim V = n$, and that V is defined by $g(x,Y) = 0$. Then $\Phi^{-1}(V)$ is defined by $g(x, \Phi_F)$, $F_i(x) = a_i x^N$, and the fiber over $b \in \mathbb{A}^n - Z$ is defined by

$$g(x, f_1(x,b) + h(b)a_1 x^N, \dots) = 0.$$

Since $b \notin Z$, $h(b) \neq 0$. Hence the left side of this equation is a nonzero polynomial in $k[x]$ for almost all choices of $a_1, \dots, a_n$ when $N >> 0$. Hence the fiber over b is finite.

The same argument works more generally when $t = m-1$; we may assume that V is defined by $g = 0$, and the fiber over b is defined by $g(X, \Phi_F(X,b)) = 0$. In order to see that the left side is nonzero for almost all choices of $F_1, \dots, F_n$ one just needs to analyze the leading form of $g(X, \Phi_F)$ with respect to X.

For any ring R we grade the polynomial ring $R[X] = R[x_1, \dots, x_m]$ with all x_i in degree 1. Any polynomial of degree d is the sum $f = F_d + \ldots + F_0$ where F_i is a homogeneous form of degree i; F_d is called the *leading form* of f with respect to X. If I is an ideal in $R[X]$ the leading forms of elements of I generate a homogeneous ideal I' of $R[X]$.

Lemma 18A.3. *Let R be a catenary Noetherian ring, $I \subset R[X]$ an ideal, and I' the ideal of leading forms in I with respect to X. Then* $\mathrm{ht}(I') = \mathrm{ht}(I)$.

Proof. Let $I_h \subset S = R[x_0, \dots, x_m]$ be the homogeneous ideal defining the closure $\bar{V}$ of $V(I)$ in $\mathbb{P}^m_R$. Then $\mathrm{ht}(I) = \mathrm{ht}_S(I_h) = \mathrm{ht}_S(I_h, x_0) - 1$. But $I' = (I_h, x_0)S/x_0S$, so $\mathrm{ht}(I') = \mathrm{ht}_S(I_h, x_0) - 1$. $\square$

Now the ring $k[X,Y]$ is bigraded, with each x_i of bidegree $(0,1)$ and each y_i of bidegree $(1,0)$. Thus each polynomial can be written as a sum $g=\sum G_{ij}$, where the G_{ij} have bidegree (i,j). Ordering the bidegrees lexicographically allows us to talk about the bidegree of g, namely the largest (p,q) with $G_{pq}\neq 0$; this G_{pq} is the bi-homogeneous leading form of g.

Without loss of generality, we assume that the generators $g_1,\dots,g_s$ of I have the following property: the bi-homogeneous leading forms $G_j(X,Y)$ of g_j generate the ideal of the leading forms of I.

Lemma 18A.4. *If $F_1,\dots,F_n$ are homogeneous forms in $k[X]$ of degree $N>\max\{\deg_X(f_i),\deg_X(g_j)\}$ then the ideal J' of leading forms in J with respect to X contains forms $h^r G_j(X,F_1,\dots,F_n)$, for $r>>0$.*

Proof. (See [HigCh] 1.6.1.) Recall that J is generated by the $g_j(X,\Phi_F)$. For any choice of the N-forms F_i it is easy to see that $\deg_X g_j(X,\Phi_F)=\deg_X G_j(X,\Phi_F) = N\deg_Y G_j+\deg_X G_j$, and that the leading form in $g_j(X,\Phi_F)$ with respect to X is $h^{\deg_Y G_j}G_j(X,F_1,\dots,F_n)$. $\square$

Proposition 18A.5. *Let $T\subset\mathbb{A}^m\times\mathbb{A}^n$ be a closed subscheme of dimension $\leq n+t$, $t\geq 0$. If k is infinite, then for any $N\geq 0$ we can find forms $F_1,\dots,F_n$ in $k[X]$ of degree N so that $W=\{w\in\mathbb{A}^m:(w,F_1(w),\dots,F_n(w))\in T\}$ has dimension at most t.*

Proof. The vector space on n-tuples $F=(F_1,\dots,F_n)$ of homogeneous forms of degree N in $k[X]$ is finite-dimensional, say dimension D. We identify it with the set of k-rational points of the affine space $\mathbb{A}^D$. Consider the evaluation map

$$\eta:\mathbb{A}^m\times\mathbb{A}^D\to\mathbb{A}^{m+n},\quad \eta(w,F)=(w,F(w)).$$

If $w\neq 0$, the fibers of $\eta: w\times\mathbb{A}^D\to w\times\mathbb{A}^n$ are isomorphic to $\mathbb{A}^{D-n}$, because the linear homomorphism $\eta(w,-):\mathbb{A}^D\to\mathbb{A}^n$ is surjective. By inspection, $\eta^{-1}(0\times\mathbb{A}^n)=0\times\mathbb{A}^D$. It follows that $\eta^{-1}(T)$ has dimension at most $D+t$.

Now consider the projection $\pi:\eta^{-1}(T)\to\mathbb{A}^D$. The theorem on dimension of the fibers [Har77] III.9.6 implies that there is a nonempty $U\subset\mathbb{A}^D$ whose fibers have dimension $\leq t$. Choosing a rational point in U, the corresponding homogeneous forms $(F_1,\dots,F_n)$ satisfy $\dim\{w\in\mathbb{A}^m:(w,F(w))\in T\}\leq t$. $\square$

Remark 18A.6. The case $N = 0$ is easy to visualize, since $D = n$. There is an open subset U of $\mathbb{A}^n$ so that for each $b \in U$ the fiber $T \cap (\mathbb{A}^m \times b)$ of the projection $T \to \mathbb{A}^n$ over b has dimension at most t.

If T is defined by bi-homogeneous polynomials, then W is defined by homogeneous polynomials. Suslin states 18A.5 for the corresponding projective varieties in [HigCh] 1.7.

We are now ready to complete the proof of theorem 18A.1. By 18A.3, $J_b \subset k[X]$ has the same height as the ideal J'_b of its leading forms. Suppose that $N > \max\{\deg_X(f_i), \deg_X(g_j)\}$. Since $h(b) \neq 0$, J'_b contains all the $G_j(X, F)$ by 18A.4. Let $T \subset \mathbb{A}^{m+n}$ be the variety defined by the ideal of bi-homogeneous forms of I, i.e., the $G_j(X, Y)$. Hence the variety $W = \{w \in \mathbb{A}^m : (w, F(w)) \in T\}$ is defined by the $G_j(X, F)$. By two applications of 18A.3, $\dim T = \dim V \leq n + t$. Thus $\dim W \leq t$ by 18A.5. But the height of J'_b is at least the height of the ideal generated by the $G_j(X, F)$, i.e., the codimension of W, which is at least $m - t$.

Lecture 19

Motivic cohomology and higher Chow groups

With the preparation of the last three lectures, we are ready to prove the fundamental comparison theorem:

Theorem 19.1. *Let X be a smooth separated scheme over a perfect field k, then for all m and i there is a natural isomorphism:*

$$H^{n,i}(X,\mathbb{Z}) \xrightarrow{\cong} CH^i(X, 2i-n).$$

Because $CH^i(X,0)$ is the classical Chow group $CH^i(X)$ we obtain:

Corollary 19.2. $H^{2i,i}(X,\mathbb{Z}) \cong CH^i(X)$

It is clear from definition 17.1 that $CH^i(X,m) = 0$ for $m < 0$. We immediately deduce the:

Vanishing Theorem 19.3. *For every smooth variety X and any abelian group A, we have $H^{n,i}(X,A) = 0$ for $n > 2i$.*

The proof of 19.1 will proceed in two stages. First we will show (in theorem 19.8) that $\mathbb{Z}(i)[2i]$ is quasi-isomorphic to $U \mapsto z^i(U \times \mathbb{A}^i, *)$ as a complex of Zariski sheaves. Then we will show (in 19.12) that the hypercohomology of $z^i(- \times \mathbb{A}^i, *)$ is $CH^i(-,*)$.

We saw in 16.7 that $\mathbb{Z}(i)$ is quasi-isomorphic to the Suslin-Friedlander motivic complex $\mathbb{Z}^{SF}(i)$. Recall from page 142 that the shift $\mathbb{Z}^{SF}(i)[2i]$ is the chain complex $C_* z_{equi}(\mathbb{A}^i, 0)$ associated to the simplicial abelian presheaf with transfers $C_\bullet z_{equi}(\mathbb{A}^i, 0)$, which sends X to $m \mapsto z_{equi}(\mathbb{A}^i, 0)(X \times \Delta^m)$. The following result generalizes example 17.2.

Lemma 19.4. *Let T be smooth of dimension d. Then for all X there is an embedding of simplicial abelian groups:*

$$C_\bullet z_{equi}(T, d-i)(X) \hookrightarrow z^i(X \times T, \bullet).$$

In particular (for $T = \mathbb{A}^i$), $\mathbb{Z}^{SF}(i)[2i](X)$ is a subcomplex of $z^i(X \times \mathbb{A}^i, *)$.

Proof. The cycles in $C_m z_{equi}(T, d-i)(X)$ are equidimensional over $X \times \Delta^m$ at all points, while the ones in $z^i(X \times T, m)$ need only be equidimensional at the generic points of the faces of $X \times T \times \Delta^m$. Hence the first group is contained in the second group of cycles. Moreover, the face maps of the two simplicial groups are compatible by 1A.12. □

Example 19.5. The complex $\mathbb{Z}^{SF}(i)[2i](Y)$ is a subcomplex of $z^i(Y \times \mathbb{A}^i, *)_{\mathcal{W}}$ (see 17.5) for every finite correspondence $\mathcal{W}$ from X to Y. Indeed, $z_{equi}(\mathbb{A}^i, 0)(Y \times \Delta^m)$ lies in $z^i(Y \times \mathbb{A}^i, m)_{\mathcal{W} \times \mathbb{A}^i}$ because every generating cycle is quasi-finite over $Y \times \Delta^m$.

In contrast, it is easy to see that $z_{equi}(Y \times \mathbb{A}^i, \dim Y)(\Delta^m)$ need not lie in $z^i(Y \times \mathbb{A}^i, m)_{\mathcal{W} \times \mathbb{A}^i}$, by letting X be a point of Y.

For any schemes X and T, consider the simplicial presheaf on X:

$$U \mapsto z^i(U \times T, \bullet).$$

This can be regarded as a simplicial sheaf on the flat site over X and hence on both the (small) étale site and the Zariski site of X as well. We will write $z^i(- \times T, *)$ for the associated complex of sheaves. The homology of $z^i(- \times T, *)$ has the more general structure of a presheaf with transfers by 17.20.

Proposition 19.6. *The homology of the embedding in 19.4 is a morphism of presheaves with transfers:*

$$H_m C_* z_{equi}(\mathbb{A}^i, 0)(-) \to H_m z^i(- \times \mathbb{A}^i, *) = CH^i(- \times \mathbb{A}^i, n). \tag{19.6.1}$$

Proof. The source and target are presheaves with transfers by 16.3 and 17.20, respectively. It suffices to show that their transfer maps are compatible.

Let W be an elementary correspondence from X to Y. We need to verify that ϕ_W and W^* are compatible with the map (19.6.1). If W is the graph of a flat map from X to Y, then ϕ_W and W^* are compatible because both are just the flat pull-back of cycles. Since W^* is defined in 17.16 by passing to an affine vector bundle torsor $Y' \to Y$, a simple diagram chase (which we leave to the reader) shows that it suffices to prove the statement when Y is affine.

Let Y be affine. Since $H_n z^i(Y \times \mathbb{A}^i, m)_W = H_n z^i(Y \times \mathbb{A}^i, m)$ by 17.5, the result will follow once we show that the following diagram commutes.

$$\begin{tikzcd} z_{equi}(\mathbb{A}^i, 0)(Y \times \Delta^m) \arrow[r, "\phi_W"] \arrow[d, hook, "19.5"'] & z_{equi}(\mathbb{A}^i, 0)(X \times \Delta^m) \arrow[d, hook, "19.4"] \\ z^i(Y \times \mathbb{A}^i, m)_W \arrow[r, "W^*"] & z^i(X \times \mathbb{A}^i, m) \end{tikzcd}$$

Let i, f and π, respectively, denote the products with $\mathbb{A}^i \times \Delta^m$ of the inclusion $W \hookrightarrow X \times Y$, and the canonical projections $X \times Y \to Y$ and $X \times Y \to X$. The transfer map W^* was defined as $W^*(\mathcal{Z}) = \pi_*((W \times \mathbb{A}^i \times \Delta^m) \cdot f^*\mathcal{Z})$ in 17.6. According to 16.3, the transfer map on $z_{equi}(\mathbb{A}^i, 0)(Y \times \Delta^m)$ is $\phi_W(\mathcal{Z}) = (i\pi)_*(\mathcal{Z}_{W \times \Delta^m})$, where the pull-back $\mathcal{Z}_{W \times \Delta^m}$ was defined on page 19. By 17A.12, $\mathcal{Z}_{W \times \Delta^m} = (fi)^*(\mathcal{Z})$, so we have:

$$\phi_W(\mathcal{Z}) = (i\pi)_*(fi)^*(\mathcal{Z}) = \pi_* i_*(fi)^*(\mathcal{Z}).$$

By 17A.11, $i_*(fi)^*(\mathcal{Z}) = (W \times \mathbb{A}^i \times \Delta^m) \cdot f^*\mathcal{Z}$ and therefore for every $\mathcal{Z}$ in $z_{equi}(\mathbb{A}^i, 0)(Y \times \Delta^m)$ we have:

$$\phi_W(\mathcal{Z}) = \pi_*((W \times \mathbb{A}^i \times \Delta^m) \cdot f^*\mathcal{Z}) = W^*(\mathcal{Z}). \qquad \square$$

Example 19.7. If E is a field over k, then the map of 19.6 evaluated at $\operatorname{Spec} E$ is an isomorphism:

$$H_m C_* z_{equi}(\mathbb{A}^i, 0)(\operatorname{Spec} E) \xrightarrow{\cong} H_m z^i(\operatorname{Spec} E \times \mathbb{A}^i, *).$$

This follows from Suslin's theorem 18.3 with $X = \mathbb{A}^i_E$, since we may identify $z_{equi}(\mathbb{A}^i_k, 0)(\Delta^m_E)$ and $z_{equi}(\mathbb{A}^i_E, 0)(\Delta^m)$ by 16.6.

This implies that theorem 19.1 is true when evaluated on fields. To see this, set $S = \operatorname{Spec} E$ and recall that $\mathbb{H}^m(S, C^*) = H^m C^*(S)$ for any complex of sheaves C^*. By 16.7, the above map fits into the sequence of isomorphisms:

$$\begin{aligned} H^{n,i}(S, \mathbb{Z}) \cong H^n\mathbb{Z}(i)(S) \cong H^n\mathbb{Z}^{SF}(i)(S) = \\ H_{2i-n} C_* z_{equi}(\mathbb{A}^i, 0)(S) \xrightarrow{\cong} H_{2i-n} z^i(\mathbb{A}^i_E, *) = \\ CH^i(\mathbb{A}^i_E, 2i-n) \cong CH^i(S, 2i-n). \end{aligned}$$

Theorem 19.8. *The map* $\mathbb{Z}^{SF}(i)[2i] = C_* z_{equi}(\mathbb{A}^i, 0) \to z^i(- \times \mathbb{A}^i, *)$ *is a quasi-isomorphism of complexes of Zariski sheaves.*

Proof. The induced homomorphisms on homology presheaves,

$$H_m C_* z_{equi}(\mathbb{A}^i, 0) \to H_m z^i(- \times \mathbb{A}^i, *) \tag{19.8.1}$$

are morphisms of presheaves with transfer by 19.6. The left side is homotopy invariant by 2.18 and the right side is homotopy invariant because the higher Chow groups are homotopy invariant (see p.149). By 19.7, this is an isomorphism for all fields. By 11.2, the sheafification of the map (19.8.1) is an isomorphism. Hence $C_* z_{equi}(\mathbb{A}^i, 0) \to z^i(- \times \mathbb{A}^i, *)$ is a quasi-isomorphism for the Zariski topology. □

Corollary 19.9. *For any smooth scheme X, the inclusion of 19.4 induces an isomorphism:*

$$H^{n,i}(X, \mathbb{Z}) \xrightarrow{\cong} \mathbb{H}^{n-2i}(X, z^i(- \times \mathbb{A}^i, *)).$$

Proof. By 16.7 and 19.8, we have the sequence of isomorphisms:

$$\begin{aligned} H^{n,i}(X, \mathbb{Z}) = \mathbb{H}^n(X, \mathbb{Z}(i)) \cong \mathbb{H}^n(X, \mathbb{Z}^{SF}(i)) = \\ \mathbb{H}^{n-2i}(X, \mathbb{Z}^{SF}(i)[2i]) \xrightarrow{\cong} \mathbb{H}^{n-2i}(X, z^i(- \times \mathbb{A}^i, *)). \end{aligned}$$ □

Corollary 19.9 is the first half of the proof of 19.1. The rest of this lecture is dedicated to proving the second half, that $\mathbb{H}^{-m}(X, z^i(- \times \mathbb{A}^i, *)) \cong CH^i(X, m)$. To do this, we shall use Bloch's Localization Theorem (see p.149) to reinterpret the higher Chow groups as the hypercohomology groups of a complex of sheaves.

A chain complex of presheaves C is said to satisfy **Zariski descent** on X if $H^*(C(U)) \to \mathbb{H}^*(U, C_{Zar})$ is an isomorphism for every open U in X.

Definition 19.10. Let C be a complex of presheaves on X_{Zar} (the small Zariski site of X). We say that C has the (Zariski) **Mayer-Vietoris property** if for every $U \subset X$, and any open covering $U = V_1 \cup V_2$, the diagram

$$\begin{array}{ccc} C(U) & \longrightarrow & C(V_1) \\ \downarrow & & \downarrow \\ C(V_2) & \longrightarrow & C(V_1 \cap V_2) \end{array}$$

is homotopy cartesian (i.e., the total complex is an acyclic presheaf). This implies that there is a long exact sequence

$$\cdots \to H^i(C(U)) \to H^i(C(V_1)) \oplus H^i(C(V_2)) \to H^i(C(V_1 \cap V_2)) \to \cdots$$

For example, any chain complex of flasque sheaves has the Mayer-Vietoris property. This is an easy consequence of the fact that $C(U) \to C(V)$ is onto for each $V \subset U$.

The following result is proven in [BG73].

Theorem 19.11 (Brown-Gersten). *Let C be a complex of presheaves on X with the Mayer-Vietoris property. Then C satisfies Zariski descent. That is, the maps $H^*(C(U)) \to \mathbb{H}^*(U, C_{Zar})$ are all isomorphisms.*

Our main application of the Brown-Gersten theorem is to prove that Bloch's complexes satisfy Zariski descent.

Proposition 19.12. *Let X be any scheme of finite type over a field. For any scheme T, each $z^i(- \times T)$ satisfies Zariski descent on X. That is, for all m and i, we have:*

$$CH^i(X \times T, m) \cong \mathbb{H}^{-m}(X, z^i(- \times T)).$$

In particular (for $T = \mathbb{A}^i$),

$$CH^i(X, m) \xrightarrow{\cong} CH^i(X \times \mathbb{A}^i, m) \xrightarrow{\cong} \mathbb{H}^{-m}(X, z^i(- \times \mathbb{A}^i)).$$

Proof. (Bloch [Blo86, 3.4]) By 19.11, we have to show that $C(U) = z^i(U \times T)$ has the Mayer-Vietoris property. For each cover $\{V_1, V_2\}$ of each U we set $V_{12} = V_1 \cap V_2$ and consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & C(U - V_1) \arrow[r] \arrow[d, "="] & C(U) \arrow[r] \arrow[d] & C(V_1) \arrow[r] \arrow[d] & \text{coker}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(V_2 - V_{12}) \arrow[r] & C(V_2) \arrow[r] & C(V_{12}) \arrow[r] & \text{coker}_2 \arrow[r] & 0.
\end{tikzcd}$$

By Bloch's Localization Theorem, the cokernels are both acyclic. A diagram chase shows that the middle square is homotopy cartesian, i.e., the Mayer-Vietoris condition is satisfied. □

We are now ready to prove the main result of this section, theorem 19.1.

Proof of 19.1. Using 19.9 and 19.12, we define the map to be the compositions of isomorphisms:

$$H^{n,i}(X,\mathbb{Z}) \cong \mathbb{H}^n(X,\mathbb{Z}(i)) \xrightarrow{\cong} \mathbb{H}^{n-2i}(X, z^i(-\times\mathbb{A}^i)) \cong CH^i(X, 2i-n). \quad \square$$

Zariski descent has also been used by Bloch and Levine to show that the higher Chow groups are functorial for morphisms between smooth schemes. We conclude this lecture by showing that their definition agrees with ours.

Definition 19.13. (Bloch-Levine) Let f be a morphism from X to Y. Natural maps $f^* : CH^i(Y,m) \to CH^i(X,m)$ for all m and i are defined as follows. As in the proof of 17.5, write $z^i(Y,*)_f$ for $z^i(Y,*)_{\Gamma_f}$.

If $U \subset Y$ is open, $z^i(Y,*)_f$ restricts to $z^i(U,*)_f$, and z^i_f is a complex of sheaves. Since Y is locally affine, $z^i_f \simeq z^i$ by 17.5 and there is a map $z^i_f \to f_* z^i$ of complexes of sheaves on Y. The map is now defined using Zariski descent 19.12 as the composite:

$$CH^i(Y,m) \cong \mathbb{H}^{-m}(Y,z^i) \cong \mathbb{H}^{-m}(Y,z^i_f) \xrightarrow{f^*} \mathbb{H}^{-m}(X,z^i) \cong CH^i(X,m).$$

Example 19.14. If $q : Y' \to Y$ is flat, then $z^i_q = z^i$, and the map q^* defined in 19.13 is just the flat pull-back of cycles map q^*, described in 17.11.

Lemma 19.15. *If $X \xrightarrow{g} Y \xrightarrow{f} Z$ are morphisms of smooth schemes, then the maps defined in 19.13 satisfy $(fg)^* = g^* f^*$.*

Proof. If $fg \coprod f : X \coprod Y \to Z$, we can restrict $(fg)^*$ and f^* to the subgroup $z^i(Z,m)_{fg\coprod f}$. Since $(fg)^* = g^* f^*$ on cycles (see [Ser65, V-30]), f^* maps this subgroup into $z^i(Y,m)_g$. By construction, the diagram of groups

$$\begin{tikzcd} z^i(Z,m)_{fg\coprod f} \arrow[r, hook] \arrow[d, "f^*"] & z^i(Z,m)_{fg} \arrow[d, "(fg)^*"] \\ z^i(Y,m)_g \arrow[r, "g^*"] & z^i(X,m) \end{tikzcd}$$

commutes. Sheafifying and applying hypercohomology, 17.5 and Zariski descent 19.12 show that the composite

$$CH^i(Z,m) \cong \mathbb{H}^{-m}(Z, z^i_{fg\coprod f}) \xrightarrow{f^*} \mathbb{H}^{-m}(Y, z^i_g) \xrightarrow{g^*} \mathbb{H}^{-m}(X, z^i) \cong CH^i(X,m)$$

is just $(fg)^*$, as required. $\square$

Proposition 19.16. *The map $f^* : CH^i(Y,m) \to CH^i(X,m)$ defined in 19.13 agrees with the map $f^* = \Gamma_f^*$ defined in 17.16.*

Proof. Suppose first that X and Y are affine, and consider the commutative diagram

$$\begin{tikzcd} CH^i(Y,m) = H^{-m}z^i(Y,*) \arrow[d, "\cong"] & H^{-m}z^i(Y,*)_f \arrow[l, "\cong"'] \arrow[r] \arrow[d] & H^{-m}z^i(X,*) = CH^i(X,m) \arrow[d, "\cong"] \\ \mathbb{H}^{-m}(Y,z^i) & \mathbb{H}^{-m}(Y,z^i_f) \arrow[l, "\cong"'] \arrow[r] & \mathbb{H}^{-m}(X,z^i). \end{tikzcd}$$

The arrows marked '$\cong$' are isomorphisms by 17.5 and 19.12. The top composite is the map of 17.11, which by 17.17 is the map Γ_f^* of 17.16. The bottom composite is the map f^* of 19.13, proving that $f^* = \Gamma_f^*$ in this case.

In the general case, 17.14 gives a diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"'] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where $X' \to X$ and $Y' \to Y$ are affine vector bundle torsors. By definition 17.16, Γ_f^* is $(p^*)^{-1}\Gamma_g^* q^*$, where p^* and q^* are flat pull-back of cycles. By 19.14, these are the same as the maps p^* and q^* defined in 19.13. Since $\Gamma_g^* = g^*$ by the first part of the proof and $g^*q^* = (qg)^* = (pf)^* = p^*f^*$ by 19.15, we have:

$$\Gamma_f^* \overset{17.16}{=} (p^*)^{-1}\Gamma_g^* q^* = (p^*)^{-1} g^* q^* \overset{19.15}{=} (p^*)^{-1} p^* f^* = f^*. \qquad \square$$

Lecture 20

Covering morphisms of triples

The main goal of the rest of the lectures will be to prove that if F is a homotopy invariant presheaf with transfers, then the presheaf $H^n_{Nis}(-,F)$ is homotopy invariant. This was stated in theorem 13.7 and it was used in lectures 13-19. The remaining lectures depend upon lectures 11, 12, and the first part of 13 (13.1–13.5), but not on the material from 13.6 to the end of lecture 19.

Definition 20.1. Let $T_Y = (\bar{Y}, Y_\infty, Z_Y)$ and $T_X = (\bar{X}, X_\infty, Z_X)$ be standard triples (as defined in 11.5). For convenience, set $Y = \bar{Y} - Y_\infty$ and $X = \bar{X} - X_\infty$. A **covering morphism** $f : T_Y \to T_X$ of standard triples is a finite morphism $f : \bar{Y} \to \bar{X}$ such that:

- $f^{-1}(X_\infty) \subset Y_\infty$ (and hence $f(Y) \subset X$);
- $f|_Y : Y \to X$ is étale;
- f induces an isomorphism $Z_Y \xrightarrow{\cong} Z_X$ with $f^{-1}(Z_X) \cap Y = Z_Y$.

Note that f need not induce a finite morphism $f : Y \to X$.

By definition, the square $Q = Q(X, Y, X - Z_X)$ induced by a covering morphism of standard triples is upper distinguished (see 12.5):

$$\begin{tikzcd} (Y - Z_Y) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ (X - Z_X) \arrow[r] & X. \end{tikzcd}$$

We say that this upper distinguished square *comes from* the covering morphism of standard triples.

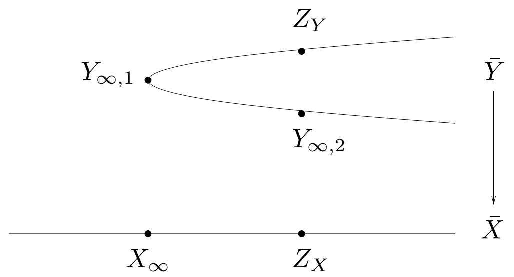

Figure 20.1: A covering morphism $f : \bar{Y} \to \bar{X}$

Example 20.2. Suppose that an affine X has a covering $X = U \cup V$ and a good compactification $(\bar{X}, X_\infty)$ over some smooth S. Then the Zariski square

$$Q(X,U,V) : \quad \begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

comes from a morphism of triples, provided that $\bar{X} - (U \cap V)$ lies in an affine open neighborhood in $\bar{X}$.

Indeed, if $Z = X - V$ then $T = (\bar{X}, X_\infty, Z)$ is a standard triple and $T' = (\bar{X}, \bar{X} - U, Z)$ is also a standard triple. The identity on $\bar{X}$ induces a covering morphism $T' \to T$ and the above square comes from this morphism.

Recall from 11.11 that a splitting of a standard triple $(\bar{X}, X_\infty, Z)$ over $V \subset X$ is a trivialization of $\mathcal{L}_{\Delta_X}$ on $V \times_S Z$.

Lemma 20.3. *Let $f : T_Y \to T_X$ be a covering morphism of standard triples. A splitting of T_X over V induces a splitting of T_Y over $f^{-1}(V) \cap Y$.*

Proof. Since T_X was split over $V \subseteq \bar{X}$, we are given $t : \mathcal{L}_{\Delta X}|_{V \times Z_X} \cong \mathcal{O}$. We need a trivialization

$$f^{-1}(t) : \mathcal{L}_{\Delta Y}|_{f^{-1}(V) \times Z_Y} \cong \mathcal{O}.$$

Now $(f \times f)^{-1}(\Delta_X)$ is the disjoint union of Δ_Y and some Q, so $(f \times f)^*(\mathcal{L}_{\Delta X})$ is $\mathcal{L}_{\Delta Y} \otimes \mathcal{L}_Q$, where $\mathcal{L}_Q$ is the associated line bundle. Since f induces an isomorphism $Z_Y \to Z_X$, Q is disjoint from $Y \times_S Z_Y$. Since $\mathcal{L}_Q$ has a canonical trivialization outside Q, we have $\mathcal{L}_Q \cong \mathcal{O}$ on $Y \times_S Z_Y$. Since $(f \times f)^*(t)$ is a trivialization of $\mathcal{L}_{\Delta_Y} \otimes \mathcal{L}_Q$ on $(f \times f)^{-1}(V \times_S Z_X)$, we may regard $(f \times f)^*(t)$ as a trivialization of $\mathcal{L}_{\Delta Y}$ on $(f^{-1}(V) \cap Y) \times_S Z_Y$. $\square$

Example 20.4. Let $\bar{Y} \to \bar{X}$ be a finite separable morphism of smooth projective curves, $X_\infty \subset \bar{X}$ a finite nonempty set containing the branch locus, and $y \in \bar{Y}$ a k-rational point so that $x = f(y)$ is not in X_∞. Set $Y_\infty = f^{-1}(X_\infty) \coprod f^{-1}(x) - \{y\}$. Then $(\bar{Y}, Y_\infty, \{y\}) \to (\bar{X}, X_\infty, \{x\})$ is a covering morphism of standard triples. If $X = \operatorname{Spec} A$ and P is the prime ideal of A defining x, then PB is prime in the coordinate ring B of Y. If $a \in A$ then by 11.13, lemma 20.3 states that if $P[1/a]$ is principal, then so is $PB[1/a]$.

Definition 20.5. Let Q be any commutative square of the form

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[d, "f"] \\ A \arrow[r, "i"] & X. \end{tikzcd}$$

We write $MV(Q)$ for the following chain complex in Cor_k:

$$MV(Q): \quad 0 \longrightarrow B \xrightarrow{(-f,i)} A \oplus Y \xrightarrow{(i,f)} X \longrightarrow 0.$$

If F is a presheaf, then $F(MV(Q))$ is the complex of abelian groups:

$$0 \longrightarrow F(X) \xrightarrow{(i,f)} F(A) \oplus F(Y) \xrightarrow{(-f,i)} F(B) \longrightarrow 0.$$

The general theorem below will involve an intricate set of data which we now describe. Let $f : (\bar{Y}, Y_\infty, Z_Y) \to (\bar{X}, X_\infty, Z_X)$ be a covering morphism of standard triples. Let Q denote the square that comes from f. Let $Q' = (X', Y', A')$ be another upper distinguished square with Y' and X' affine so that Q and Q' are of the form:

$$Q': \quad \begin{tikzcd} B' \arrow[r] \arrow[d, "f'"] & Y' \arrow[d, "f'"] \\ A' \arrow[r, "i'"] & X' \end{tikzcd} \qquad Q: \quad \begin{tikzcd} B \arrow[r] \arrow[d, "f"] & Y \arrow[d, "f"] \\ A \arrow[r, "i"] & X. \end{tikzcd} \tag{20.5.1}$$

Theorem 20.6. *Let $j : Q' \to Q$ be a morphism of upper distinguished squares of the form 20.5.1 such that:*

- *Q comes from a covering morphism of standard triples;*
- *$X' \to X$ is an open embedding, and $(\bar{X}, X_\infty, Z_X)$ splits over X';*
- *X' and Y' are affine.*

Then for any homotopy invariant presheaf with transfers F, the map of complexes $F(MV(Q)) \to F(MV(Q'))$ is chain homotopic to zero.

$$\begin{tikzcd} 0 \arrow[r] & F(X) \arrow[r, "{(i,f)}"] \arrow[d, "j_X"'] & F(A) \oplus F(Y) \arrow[r, "{(-f,i)}"] \arrow[d, "\binom{j_A}{j_Y}"] & F(B) \arrow[r] \arrow[d, "j_B"] & 0 \\ 0 \arrow[r] & F(X') \arrow[r, "{(i',f')}"] & F(A') \oplus F(Y') \arrow[r, "{(-f',i')}"] & F(B') \arrow[r] & 0 \end{tikzcd}$$

The proof will be assembled from the following three lemmas.

We say that a diagram in Cor_k is *homotopy commutative* if every pair of composites $f, g : X \to Y$ with the same source and target are $\mathbb{A}^1$-homotopic. Any homotopy invariant presheaf with transfers identifies $\mathbb{A}^1$-homotopic maps, and converts a homotopy commutative diagram into a commutative diagram.

Lemma 20.7. *Let $j : Q' \to Q$ be as in the statement of 20.6. Then there are maps $\lambda_A \in Cor(X', A)$ and $\lambda_B \in Cor(Y', B)$, well-defined up to $\mathbb{A}^1$-homotopy, such that the following diagram is homotopy commutative.*

$$\begin{tikzcd} & Y' \arrow[r, "f'"] \arrow[dl, "j_Y"'] \arrow[d, "\exists\lambda_B"] & X' \arrow[d, "\exists\lambda_A"'] \arrow[dr, "j_X"] & \\ Y & B \arrow[l, "i"'] \arrow[r, "f"] & A \arrow[r, "i"] & X. \end{tikzcd}$$

Applying a homotopy invariant presheaf with transfers F gives a commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r, "i"] \arrow[dr, "j_X"'] & F(A) \arrow[r, "f"] \arrow[d, "\exists\lambda_A"] & F(B) \arrow[d, "\exists\lambda_B"'] & F(Y) \arrow[l, "i"'] \arrow[dl, "j_Y"] \\ & F(X') \arrow[r, "f'"] & F(Y'). & \end{tikzcd}$$

Proof. By 20.3, both triples T_X and T_Y split. Hence the maps in question exist and the outer triangles commute by 11.15. The construction of the trivializations in the proof of 11.15 shows that the middle square commutes. $\square$

Since $H_0^{sing}(X \times Y/X) = Cor(X, Y)/\mathbb{A}^1$-homotopy by 7.2, two elements of $Cor(X, Y)$ are $\mathbb{A}^1$-homotopic exactly when they agree in $H_0^{sing}(X \times Y/X)$. This allows us to apply the techniques of lecture 7.

Lemma 20.8. *Let h be a rational function on $\bar{X} \times_S \bar{Y}$ which is invertible in a neighborhood U of $A' \times_S Y_\infty$ and $A' \times_S Z_Y$, and equals 1 on $A' \times_S Y_\infty$. Then the Weil divisor D defined by h defines an element ψ of $Cor(A', B)$ such that the composition $i\psi \in Cor(A', Y)$ is $\mathbb{A}^1$-homotopic to zero.*

Proof. As a divisor on the normal variety $A' \times_S \bar{Y}$, we can write $D = \sum n_i D_i$ with each D_i integral and supported off of U. Since each D_i misses $A' \times_S Y_\infty$, it is quasi-finite over A'. Since D_i is proper over A', and has the same dimension as A', it is finite and surjective over A'. As such, each D_i and hence D defines an element of $C_0(A' \times_S B/A')$ which is a subgroup of $C_0(A' \times B/A') = Cor(A', B)$. By construction (see 7.15), the image of D in $\mathrm{Pic}(A' \times_S \bar{Y}, A' \times_S (Y_\infty \coprod Z))$ is given by $(\mathcal{O}, h)$, the trivial line bundle with trivialization 1 on $A' \times_S Y_\infty$, and h on $A' \times_S Z_Y$. The composition with $i : B \to Y$ sends D to an element of $C_0(A' \times_S Y/A')$ whose image in $\mathrm{Pic}(A' \times_S \bar{Y}, A' \times_S Y_\infty)$ is the class of $(\mathcal{O}, h)$. By 7.16, this group is isomorphic to $H_0^{sing}(A' \times_S B/A')$. But in this group $(\mathcal{O}, h) = (\mathcal{O}, 1)$ is the zero element. This implies that the image is zero in $H_0^{sing}(A' \times B/A')$. $\square$

Lemma 20.9. *Let $j : Q' \to Q$ be as in the statement of 20.6, and λ_A, λ_B as in 20.7. Then there is a map ψ in $Cor_k(A', B)$ fitting into a homotopy commutative diagram:*

$$\begin{tikzcd}
B' \arrow[r, "\lambda_B \circ i' - j_B"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\
A' \arrow[r, "\lambda_A \circ i' - j_A"'] \arrow[ur, dotted, "\psi"] & A.
\end{tikzcd}$$

Moreover the composition $A' \xrightarrow{\psi} B \xrightarrow{i} Y$ is $\mathbb{A}^1$-homotopic to 0.

Applying a homotopy invariant presheaf with transfers F gives a commutative diagram:

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "i' \circ \lambda_A - j_A"'] & F(B) \arrow[d, "i' \circ \lambda_B - j_B"] \arrow[dl, dotted, "\psi"'] \\ F(A') \arrow[r, "f'"'] & F(B'), \end{tikzcd}$$

and the composite $F(Y) \xrightarrow{i} F(B) \xrightarrow{\psi} F(A')$ *is zero.*

Proof of 20.9. In order to streamline notation, we write $\times$ for $\times_S$.

Let $\mathcal{L}_{\Delta X'}$ be the line bundle on $X' \times \bar{X}$ corresponding to the graph $\Delta X'$ of $X' \hookrightarrow \bar{X}$, and $\mathcal{L}_{\Delta Y'}$ for the line bundle on $Y' \times \bar{Y}$ corresponding to the graph $\Delta Y'$ of $Y' \hookrightarrow \bar{Y}$. In between these, we have the line bundle $\mathcal{M}$ on $X' \times \bar{Y}$, obtained by pulling back $\mathcal{L}_{\Delta X'}$.

Since these three line bundles come from effective divisors, they have canonical global sections. We will write s_X for the canonical global section of $\mathcal{L}_{\Delta X'}$ on $X' \times \bar{X}$, $s_{\mathcal{M}}$ for $\mathcal{M}$ on $X' \times \bar{Y}$, and s_Y for $\mathcal{L}_{\Delta Y'}$ on $Y' \times \bar{Y}$. Each global section determines a section on $X' \times Z_X$, $X' \times Z_Y$, and $Y' \times Z_Y$, respectively. Since $A' \subseteq X' - Z_X$ and $B' \subseteq Y' - Z_Y$, the restrictions of $s_X, s_{\mathcal{M}}, s_Y$ also determine trivializations in each case, of $\mathcal{L}_{\Delta X'}$ on $A' \times Z_X$, of $\mathcal{M}$ on $A' \times Z_Y$, and of $\mathcal{L}_{\Delta Y'}$ on $B' \times Z_Y$.

Because $Z_Y \cong Z_X$, the inclusion of $X' \times Z_X$ in $X' \times \bar{X}$ lifts to $X' \times \bar{Y}$, and we may identify the pullbacks of $\mathcal{L}_{\Delta X'}$ and $\mathcal{M}$ to $X' \times Z_Y$, together with their respective trivializations s_X and $s_{\mathcal{M}}$ on $A' \times Z_Y$.

Since the standard triple $(\bar{X}, X_\infty, Z_X)$ splits over X', we are given a fixed trivialization t_X of $\mathcal{L}_{\Delta X'}$ on $X' \times Z_X$. As with s_X, we may identify t_X with a trivialization $t_{\mathcal{M}}$ of $\mathcal{M}$ on $X' \times Z_Y$. By 20.3, t_X also induces a trivialization t_Y of $\mathcal{L}_{\Delta Y}$ on $Y' \times Z_Y$. Since Z_X lives in an affine neighborhood U_X in $\bar{X}$, we extend t_X to $X' \times U_X$ and we fix this particular extension. Pulling back, the same is true for $t_{\mathcal{M}}$ and t_Y and we fix those two extensions too.

Because $t_X, t_{\mathcal{M}}, t_Y$ are trivializations, there are regular functions $r_X, r_{\mathcal{M}}, r_Y$ so that:

$$s_X = r_X t_X \text{ on } X' \times Z_X; \quad s_{\mathcal{M}} = r_{\mathcal{M}} t_{\mathcal{M}} \text{ on } X' \times Z_Y; \quad s_Y = r_Y t_Y \text{ on } Y' \times Z_Y.$$

Because s_X is a trivialization on $A' \times Z_X$, r_X is invertible on $A' \times Z_X$. Similarly, $r_{\mathcal{M}}$ is invertible on $A' \times Z_Y$, and r_Y is invertible on $B' \times Z_Y$. (See figure 20.2.)

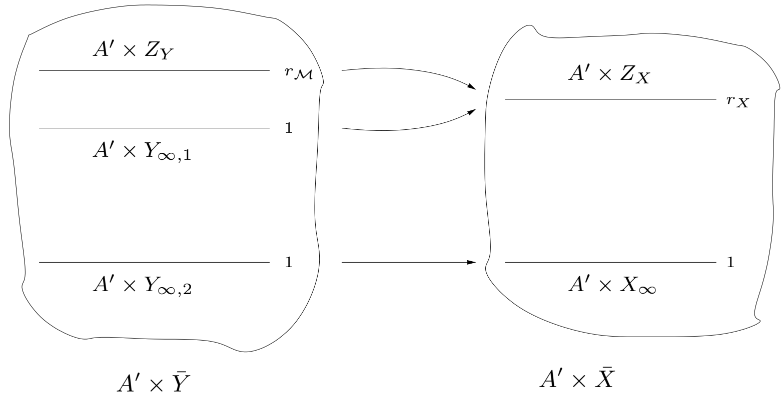

Figure 20.2: The covering morphism $f : \bar{Y} \to \bar{X}$ over A'

Because $(\bar{Y}, Y_\infty, Z_Y)$ is a standard triple, there is an affine open neighborhood U of $Y_\infty \coprod Z_Y$ in $\bar{Y}$. Hence $X' \times U$ is an affine open neighborhood of $X' \times Z_Y$ and $X' \times Y_\infty$ in $X' \times \bar{Y}$. Since Z_Y and Y_∞ are disjoint, the Chinese Remainder Theorem yields a regular function h on $X' \times U$ which equals 1 on $X' \times Y_\infty$ and equals $r_{\mathcal{M}}$ on $X' \times Z_Y$. Let $D \subset X' \times \bar{Y}$ denote the the principal divisor corresponding to h. By lemma 20.8, the divisor $-D$ defines an element ψ of $Cor(A', B)$ such that the composition $i\psi \in Cor(A', Y)$ is homotopically trivial. By 7.15, the map $Cor(A', B) \to \mathrm{Pic}(A' \times \bar{Y}, A' \times (Y_\infty \coprod Z_Y))$ sends ψ to the class of $(\mathcal{O}_{A' \times \bar{Y}}, 1_\infty \coprod r_{\mathcal{M}}^{-1})$.

It remains to verify that the diagram in 20.9 is homotopy commutative.

We first interpret the horizontal maps in 20.9. By the construction of λ_A and λ_B in 11.15 and 20.7, the compositions $\lambda_A \circ i' \in Cor(A', A)$ and $\lambda_B \circ i' \in Cor(B', B)$ represent the classes of $(\mathcal{L}_{\Delta A'}, s_\infty \coprod t_X)$ and $(\mathcal{L}_{\Delta B'}, s_\infty \coprod t_Y)$ in $\mathrm{Pic}(A' \times \bar{X}, A' \times (X_\infty \coprod Z_Y))$ and $\mathrm{Pic}(B' \times \bar{Y}, B' \times (Y_\infty \coprod Z_Y))$, respectively. On the other hand, the inclusions j_A and j_B represent the classes of $(\mathcal{L}_{\Delta A'}, s_\infty \coprod s_X)$ and $(\mathcal{L}_{\Delta B'}, s_\infty \coprod s_Y)$, respectively. It follows that the differences $j_A - \lambda_A \circ i' \in Cor(A', A)$ and $j_B - \lambda_B \circ i' \in Cor(B', B)$ represent the classes of $(\mathcal{O}_{A' \times \bar{X}}, 1_\infty \coprod r_X)$ and $(\mathcal{O}_{B' \times \bar{Y}}, 1_\infty \coprod r_Y)$, respectively. (Cf. exercise 11.16.)

The composition $\psi f' \in Cor(B', B)$ represents $(\mathcal{O}_{B' \times \bar{Y}}, f^* h^{-1})$. Since $f^* h$ is a rational function on $B' \times \bar{Y}$ which is 1 on $B' \times Y_\infty$ and r_Y on $B' \times Z_Y$, we have $\psi f' = \lambda_B \circ i' - j_B$ in $\mathrm{Pic}(B' \times \bar{Y}, B' \times (Y_\infty \coprod Z_Y))$.

Now the composition $f\psi \in Cor(A', A)$ represents the push-forward of ψ along $H_0(A' \times B/A') \to H_0(A' \times A/A')$. By 7.24, this represents the class of $(\mathcal{O}_{A' \times \bar{X}}, f_*(1_\infty \coprod r_{\mathcal{M}}^{-1}))$. By definition 7.22, the norm of h is a rational function which extends the trivialization $f_*(1_\infty \coprod r_{\mathcal{M}})$ to an affine neighborhood. Since h is identically 1 on $f^{-1}(X_\infty) \subset Y_\infty$, $N(h) = 1$ on $A' \times X_\infty$ by 7.23. We will show that $N(h) = r_X$ on $A' \times Z_X$ in lemma 20.10 below. Hence $f\psi = \lambda_A i' - j_A$ in $Cor(A', A)$, as desired. $\square$

Lemma 20.10. *Let $f : U \to V$ be a finite map with U and V normal. Suppose that $Z \subset V$ and $Z' \subset U$ are reduced closed subschemes such that the induced map $Z' \to Z$ is an isomorphism, and $U \to V$ is étale in a neighborhood of Z'.*

If $h \in \mathcal{O}^*(U)$ is 1 on $f^{-1}(Z) - Z'$, then $N(h)|_Z$ and $h|_{Z'}$ are identified by $Z' \cong Z$.

Proof. Suppose first that f has a section $s : V \to U$ sending Z to Z'. Then $U \cong s(U) \coprod U'$ and h is 1 on $f^{-1}(Z) \cap U'$. In this case, the assertion follows from the componentwise calculation of the norm $N(h)$, together with 7.23.

In the general case, let $U' \subset U$ be a neighborhood of Z' which is étale over V, and let $h' \in \mathcal{O}^*(U' \times_V U)$ be the pullback of h. The graph $Z'' \subset U' \times_V U$ of $Z' \to Z$ is isomorphic to Z', and $U' \times_V U'$ is an étale neighborhood of Z'' in $U' \times_V U$. By construction, h' is 1 on $U' \times_V (f^{-1}(Z) - Z'')$ and $U' \times_V U \to U'$ has a canonical section sending Z' to Z''; in this case we have shown that $N(h')|_{Z'}$ is identified with $h|_{Z'}$. Since norms commute with base change, we can identify $N(h)$ with $N(h')$ under $\mathcal{O}^*(V) \subseteq \mathcal{O}^*(U')$. This proves the lemma. $\square$

Proof of 20.6. From 20.7 and 20.8, we have maps $s_1 = (\lambda_A, 0) : F(A) \oplus F(Y) \to F(X')$ and $s_2 = (\psi, \lambda_B) : F(B) \to F(A') \oplus F(Y')$. In order for these maps to form a chain homotopy from j to zero, we must have $sd + ds = j$. This amounts to six equations, three of which come from the commutativity of the trapezoid in 20.7. The other three, which involve ψ are: $\psi i \simeq 0$, $j_A \simeq i'\lambda_A - \psi f$ and $j_B \simeq i'\lambda_B - f'\psi$. These are provided by 20.9. $\square$

Corollary 20.11. *Let $Q = Q(X, Y, A)$ be an upper distinguished square of smooth schemes coming from a covering morphism of standard triples and let $S = \{x_1, \ldots, x_n\}$ be a finite set of points in X.*

Then there exists an affine neighborhood X' of S in X such that:

- *The induced square* $Q' = Q(X', Y', A')$ *is upper distinguished, where* $A' = X' \cap A$ *and* $Y' = f^{-1}(X')$*;*

- *For any homotopy invariant presheaf with transfers* F*, the map* $F(MV(Q)) \to F(MV(Q'))$ *is chain homotopic to zero.*

$$\begin{tikzcd}
0 \arrow[r] & F(X) \arrow[r] \arrow[d] & F(A) \oplus F(Y) \arrow[r] \arrow[d] & F(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(X') \arrow[r] & F(A') \oplus F(Y') \arrow[r] & F(B') \arrow[r] & 0
\end{tikzcd}$$

Proof. By 11.14, S has an affine neighborhood X' over which the triple $(\bar{X}, X_\infty, Z)$ splits. We may shrink X' in order to assume that it is disjoint from $f(Y_\infty)$. Hence $Y' = f^{-1}(X')$ is finite over X', and hence affine, as well as being contained in Y. The induced square Q' is upper distinguished because it is obtained from Q by base change. Thus the hypotheses of theorem 20.6 are satisfied for $Q' \to Q$. $\square$

Lecture 21

Zariski sheaves with transfers

With the technical results of the last lecture in hand, we are ready to prove the following results.

Theorem 21.1. *Let F be a homotopy invariant presheaf with transfers. Then the Zariski sheaf F_{Zar} is homotopy invariant.*

Theorem 21.2. *Let F be a homotopy invariant presheaf with transfers. Then $F_{Zar} = F_{Nis}$.*

Combining 21.1 and 21.2, we obtain theorem 21.3 below, which is the case $n = 0$ of theorem 13.7. This theorem does not require k to be perfect.

Theorem 21.3. *If F is a homotopy invariant presheaf with transfers, then the Nisnevich sheaf F_{Nis} is homotopy invariant.*

We will prove theorems 21.1 and 21.2 in order, using a sequence of lemmas. We make the running assumption that F is a homotopy invariant presheaf with transfers. The Mayer-Vietoris sequence $F(MV(Q))$ associated to a commutative square Q is defined in 20.5.

Lemma 21.4. *Let U be an open subset of $\mathbb{A}^1$ and $U = U_1 \cup U_2$ be a Zariski covering of U. Then the complex $F(MV(Q))$ is split exact, where $Q = Q(U, U_1, U_2)$.*

$$F(MV(Q)): \quad 0 \longrightarrow F(U) \longrightarrow F(U_1) \oplus F(U_2) \longrightarrow F(U_1 \cap U_2) \longrightarrow 0$$

In particular, F is a Zariski sheaf on $\mathbb{A}^1$.

Proof. Setting $Y_\infty = \mathbb{P}^1 - U$, $Y'_\infty = \mathbb{P}^1 - U_1$ and $Z = U - U_2$, the identity of $\mathbb{P}^1$ is a covering morphism $(\mathbb{P}^1, Y'_\infty, Z) \to (\mathbb{P}^1, Y_\infty, Z)$ of standard triples as in example 20.2. Both triples are split over U itself by 11.13, so by theorem 20.6 with $Q' = Q$, the complex $F(MV(Q))$ is chain contractible, i.e., split exact. □

Lemma 21.5. *If F is a homotopy invariant Zariski sheaf with transfers, and U is an open subset of $\mathbb{A}^1$, then $H^n_{Zar}(U, F) = 0$ for $n > 0$.*

Proof. If $\mathcal{U} = \{U_1, \dots, U_n\}$ is a finite cover of U, it follows from 21.4 and induction on n that the following sequence is exact.

$$0 \to F(U) \to \oplus_i F(U_i) \to \oplus_{i,j} F(U_i \cap U_j) \to \dots \to F(\cap_i U_i) \to 0$$

Hence the Čech cohomology of F satisfies $\check{H}^i(\mathcal{U}, F) = 0$ for $i > 0$. But then $H^1(U, F) = \check{H}^1(U, F) = 0$ by [Har77, Ex III.4.4]. Since $\dim U = 1$, we must also have $H^i(U, F) = 0$ for $i > 1$ (see [Har77, III.2.7]). □

Exercise 21.6. Show that 21.4 and 21.5 fail for $F = \mathcal{O}^*_X$ if $\mathbb{A}^1$ is replaced by an affine elliptic curve.

Lemma 21.7. *If F is a homotopy invariant Nisnevich sheaf with transfers, and U is an open subset of $\mathbb{A}^1$, then $H^n_{Nis}(U, F) = 0$ for $n > 0$.*

Proof. Since $\dim U = 1$, we have $H^n_{Nis}(U, F) = 0$ for $n > 1$. By [Mil80, III.2.10], $H^1_{Nis}(U, F) = \check{H}^1(U, F)$. Therefore we only need to show that $\check{H}^1(U, F) = 0$.

Since F takes disjoint unions to direct sums, the Čech cohomology can be computed using covering families $V \to X$, instead of the more general $\{V_i \to X\}$. By 12.6, any such cover of U has a refinement $\mathcal{U} = \{A, V\}$, where $A \subset U$ is dense open, $V \to U$ is étale, and the square $Q = Q(U, V, A)$ is upper distinguished (see 12.5). Embed V in a smooth projective curve $\bar{V}$ finite over $\mathbb{P}^1$, and set $V_\infty = \bar{V} - V$. By construction (see 20.1), Q comes from the covering morphism of standard triples $(\bar{V}, V_\infty, Z) \to (\mathbb{P}^1, U_\infty, Z)$, where $U_\infty = \mathbb{P}^1 - U$ and $Z = U - A$. Since $(\mathbb{P}^1, U_\infty, Z)$ splits over U by 11.13, theorem 20.6 with $Q' = Q$ implies that the complex $F(MV(Q))$ is split exact. That is $\check{H}^1(\mathcal{U}, F) = 0$. Passing to the limit over all such covers yields $\check{H}^1(U, F) = 0$. □

Lemma 21.8. *Let F be a homotopy invariant presheaf with transfers. If X is smooth and $U \subset X$ is dense open, then $F_{Zar}(X) \to F_{Zar}(U)$ is injective.*

Proof. As F_{Zar} is a sheaf it suffices to verify this locally. Let $f \in F_{Zar}(X)$ be a nonzero section which vanishes in $F_{Zar}(U)$. Pick a point $x \in X$ so that f is nonzero in the stalk $F_x = F(\operatorname{Spec} \mathcal{O}_{X,x})$. By shrinking X around x we may assume that $f \in F(X)$. By shrinking U, we may assume that f vanishes in $F(U)$ and hence in $F(V)$ for $V = \operatorname{Spec}(\mathcal{O}_{X,x}) \cap U$. By 11.1, f is nonzero in $F(V)$, and this is a contradiction. □

Proof of 21.1. We have to prove that $i^* : F_{Zar}(X \times \mathbb{A}^1) \to F_{Zar}(X)$ is an isomorphism, where $i : X \to X \times \mathbb{A}^1$. It is enough to prove that i^* is injective. We may assume that X is connected and therefore irreducible. Let $\gamma : \operatorname{Spec} K \longrightarrow X$ be the generic point. We get a diagram:

$$\begin{tikzcd} F_{Zar}(X \times \mathbb{A}^1) \arrow[r, "i^*"] \arrow[d, "(\gamma \times 1)^*"'] & F_{Zar}(X) \arrow[d, "\gamma^*"] \\ F_{Zar}(\operatorname{Spec} K \times \mathbb{A}^1) \arrow[r, "\cong"] & F_{Zar}(\operatorname{Spec} K) \end{tikzcd}$$

where the vertical maps are injective by 21.8. The bottom map is an isomorphism by 21.4 since we may regard F as a homotopy invariant presheaf with transfers over the field K by 2.9:

$$F_{Zar}(\mathbb{A}^1_K) = F(\mathbb{A}^1_K) \xrightarrow{\cong} F(\operatorname{Spec} K) = F_{Zar}(\operatorname{Spec} K).$$

Thus i^* is injective. □

Let $s_{Zar}(F)$ be the separated presheaf (with respect to the Zariski topology) associated to the presheaf F. It is defined by the formula:

$$s_{Zar}(F)(X) = F(X)/F_0(X), \quad F_0(X) = \operatorname*{colim}_{\substack{\text{covers} \\ \{U_i \to X\}}} \ker F(X) \to \prod F(U_i).$$

Lemma 21.9. *$s_{Zar}(F)$ is a homotopy invariant presheaf with transfers.*

Proof. The homotopy invariance of $s_{Zar}F$ is immediate from the fact that homotopy invariance is preserved by quotient presheaves. The existence of transfers is more difficult. Let $Z \subset S \times X$ be an elementary correspondence from S to X. We must show that the corresponding transfer $F(X) \to F(S)$ sends $F_0(X)$ to $F_0(S)$, i.e., that the image of $F_0(X)$ vanishes at each stalk $F(\operatorname{Spec} \mathcal{O}_{S,s})$. It suffices to suppose S local, so that Z is semilocal. Hence there is a semilocal subscheme X' of X with $Z \subset S \times X'$. But by 11.1, $F(X')$ injects into $F(U)$ for each dense $U \subset X'$, so $F_0(X') = 0$. Hence $F_0(X) \to F(S)$ is zero, because it factors through $F_0(X') = 0$. □

For the next few lemmas, S will be the semilocal scheme of a smooth quasi-projective variety X at a finite set of points. Since any finite set of points lies in an affine neighborhood, we may even assume that X is affine. Clearly, S is the intersection of the filtered family of its affine open neighborhoods X_α in X.

Lemma 21.10. *Suppose that F is a homotopy invariant presheaf with transfers. Then for any open covering $S = U_0 \cup V$ there is an open $U \subset U_0$ such that $S = U \cup V$ and the sequence $F(MV(Q))$ is exact, where $Q = Q(S, U, V)$:*

$$0 \to F(S) \to F(U) \oplus F(V) \to F(U \cap V) \to 0.$$

Proof. We may assume that S is connected, since we can work separately with each component. By assumption, there are open $\tilde{U}_0$, $\tilde{V}$ in X such that $U_0 = S \cap \tilde{U}_0$, $V = S \cap \tilde{V}$. Since $\tilde{U}_0$ is open in X, there is an affine open $\tilde{U}$ contained in $\tilde{U}_0$ which contains the finite set of closed points of U_0. Setting $U = S \cap \tilde{U}$, we have $S = U \cup V$. We will show that $F(MV(Q))$ is exact for the square $Q = Q(S, U, V)$.

We first suppose that k is an infinite field. For each α, set $U_\alpha = X_\alpha \cap \tilde{U}$ and $V_\alpha = X_\alpha \cap \tilde{V}$. The canonical map from Q to the square $Q_\alpha = Q(X_\alpha, U_\alpha, V_\alpha)$ induces a morphism of Mayer-Vietoris sequences, $F(MV(Q_\alpha)) \to F(MV(Q))$. It suffices to show that these morphisms are chain homotopic to zero, because $F(MV(Q))$ is the direct limit of the $F(MV(Q_\alpha))$.

Let $Z \subset X$ denote the union of $X - (\tilde{U} \cap \tilde{V})$ and the closed points of S. For each X_α, we know by 11.17 that there is an affine neighborhood X'_α of S in X_α and a standard triple $T_\alpha = (\bar{X}_\alpha, X_{\infty,\alpha}, Z_\alpha)$ with $X'_\alpha \cong \bar{X}_\alpha - X_{\infty,\alpha}$ and $Z_\alpha = X_\alpha \cap Z$. Set $U'_\alpha = X'_\alpha \cap \tilde{U}$ and $V'_\alpha = X'_\alpha \cap \tilde{V}$. Since $\bar{X}_\alpha - (U'_\alpha \cap V'_\alpha)$ lies in $X_{\infty,\alpha} \cup Z_\alpha$, it lies in an affine open subset of $\bar{X}_\alpha$ (by definition 11.5). By 20.2, the Zariski square $Q'_\alpha = Q(X'_\alpha, U'_\alpha, V'_\alpha)$ comes from a covering morphism of triples $T'_\alpha \to T_\alpha$.

By 11.14, the triple T_α is split over an affine neighborhood X''_α of S in X'_α. Set $U''_\alpha = X''_\alpha \cap \tilde{U}$ and $V''_\alpha = X''_\alpha \cap \tilde{V}$, and form the square $Q''_\alpha = Q(X''_\alpha, U''_\alpha, V''_\alpha)$. Since X''_α and $\tilde{U}$ are affine, so is U''_α. By theorem 20.6, the morphism $F(MV(Q'_\alpha)) \to F(MV(Q''_\alpha))$ is chain homotopic to zero. Since $F(MV(Q_\alpha)) \to F(MV(Q))$ factors through this morphism, it too is chain homotopic to zero.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X_\alpha) & \longrightarrow & F(U_\alpha)\oplus F(V_\alpha) & \longrightarrow & F(U_\alpha\cap V_\alpha) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X'_\alpha) & \longrightarrow & F(U'_\alpha)\oplus F(V'_\alpha) & \longrightarrow & F(U'_\alpha\cap V'_\alpha) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X''_\alpha) & \longrightarrow & F(U''_\alpha)\oplus F(V''_\alpha) & \longrightarrow & F(U''_\alpha\cap V''_\alpha) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(S) & \longrightarrow & F(U)\oplus F(V) & \longrightarrow & F(U\cap V) & \longrightarrow & 0.
\end{array}$$

If k is finite, exactness follows by a transfer argument. Any element a in the homology of $F(MV(Q))$ must vanish when we pass to $Q\otimes_k k'$ for any infinite algebraic extension k' of k. Since a must vanish for some finite subextension k'_0, a has exponent $[k'_0 : k]$. Since $[k'_0 : k]$ can be chose to be a power of any prime, we conclude that $a = 0$. $\square$

Lemma 21.10 corrects [CohTh, 4.23], which omitted the passage from U_0 to U.

Corollary 21.11. *Let S' and S'' be semilocal schemes of a smooth quasi-projective scheme X at finite sets of points, and set $S = S'\cup S''$. Then the Mayer-Vietoris sequence $F(MV(Q))$ is exact, where $Q = Q(S, S', S'')$:*

$$0 \to F(S) \to F(S')\oplus F(S'') \to F(S'\cap S'') \to 0.$$

Proof. Write S' as the intersection of opens $U_\alpha\subset S$ and S'' as the intersection of opens $V_\beta\subset S$. The sequence $F(MV(Q))$ is the direct limit of the sequences $F(MV(Q_{\alpha\beta}))$, where $Q_{\alpha\beta} = Q(S, U_\alpha, V_\beta)$. By 21.10, there are $U_{\alpha\beta}\subset U_\alpha$ such that the sequences $F(MV(Q(S, U_{\alpha\beta}, V_\beta)))$ are exact. Hence the morphisms from $F(MV(Q_{\alpha\beta}))$ to $F(MV(Q))$ are zero on homology. Passing to the direct limit, we see that the homology of $F(MV(Q))$ is zero, i.e., it is exact. $\square$

Note that the sequence $0 \to \mathcal{F}(S) \to \mathcal{F}(S')\oplus\mathcal{F}(S'') \to \mathcal{F}(S'\cap S'')$ is always exact when $\mathcal{F}$ is a Zariski sheaf on S. This is because it is the direct limit of the exact sequences $0 \to \mathcal{F}(S) \to \mathcal{F}(U_\alpha)\oplus\mathcal{F}(V_\beta) \to \mathcal{F}(U_\alpha\cap V_\beta)$ associated to the family of open covers $\{U_\alpha, V_\beta\}$ of S with $S'\subset U_\alpha$ and $S''\subset V_\beta$.

Lemma 21.12. *Let S be the semilocal scheme of a smooth quasi-projective scheme X at a finite set of points. Then $F_{Zar}(S) = F(S)$.*

Proof. By 11.1, $F(S) = (s_{Zar}F)(S)$. Since $s_{Zar}F$ is a homotopy invariant presheaf with transfers by 21.9, we may replace F by $s_{Zar}F$ and assume that F is separated. We now proceed by induction on the number of the closed points of S. Let S' be the local scheme at a closed point x of S, and S'' the semilocal scheme at the remaining points. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & F(S) \arrow[r] \arrow[d] & F(S') \oplus F(S'') \arrow[r] \arrow[d, "="] & F(S' \cap S'') \arrow[d, "\text{into}"] \\
0 \arrow[r] & F_{Zar}(S) \arrow[r] & F_{Zar}(S') \oplus F_{Zar}(S'') \arrow[r] & F_{Zar}(S' \cap S'')
\end{tikzcd}$$

The top row is exact by 21.11, and we have noted that the bottom row is exact because F_{Zar} is a Zariski sheaf. The right vertical map is an injection because F is separated. The middle vertical map is the identity by induction. A diagram chase shows that the left vertical map is an isomorphism, as desired. ☐

We need an analogue of lemma 6.16 for the Zariski topology, showing that we can lift finite correspondences to open covers under mild conditions.

Lemma 21.13. *Let W be a closed subset of $X \times Y$, $x \in X$ a point and $V \subset Y$ an open subset such that $p^{-1}(x) \subset \{x\} \times V$, where $p : W \to X$ is the projection. Then there is a neighborhood U of x such that $W \times_X U$ is contained in $U \times V$.*

Proof. The subset $Z = W - W \cap (X \times V)$ is closed, and $x \notin p(Z)$. Because p is a closed map, $p(Z)$ is closed and $U = X - p(Z)$ is an open neighborhood of x. By construction, $W \times_X U$ is contained in $U \times V$. ☐

Corollary 21.14. *Let $\mathcal{W} \in Cor(X, Y)$ have support W and let $p : W \to X$ be the projection. If $x \in X$ and $V \subset Y$ are such that $p^{-1}(x) \subset \{x\} \times V$, then there is a neighborhood U of x and a canonical $\mathcal{W}_U \in Cor(U, V)$ such that the following diagram commutes.*

$$\begin{tikzcd}
U \arrow[r, "\mathcal{W}_U"'] \arrow[d] & V \arrow[d] \\
X \arrow[r, "\mathcal{W}"] & Y
\end{tikzcd}$$

Proof. Writing $\mathcal{W} = \sum n_i[W_i]$, we may apply lemma 21.13 to each W_i. Since W_i is finite over X, $W_i \times_X U$ is finite over U, so $\mathcal{W}_U = \sum n_i[W_i \times_X U]$ is the required finite correspondence. It is canonical because if $U' \subset U$, the composition of $U' \subset U$ with $\mathcal{W}_U$ is $\mathcal{W}_{U'} = \sum n_i[W_i \times_X U']$. $\square$

Theorem 21.15. *Let F be a homotopy invariant presheaf with transfers. Then the Zariski sheaf F_{Zar} has a unique structure of presheaf with transfers such that $F \to F_{Zar}$ is a morphism of presheaves with transfers.*

Proof. By 21.9 we may assume that F is separated, i.e., that $F(V) \subseteq F_{Zar}(V)$ for every V. We may also assume that X and Y are irreducible without loss of generality.

We begin with a general construction, starting with an element $f \in F_{Zar}(Y)$ and a finite correspondence $\mathcal{W}$ from X to Y. Fix a point $x \in X$. Since $p : W \to X$ is finite, the image of $p^{-1}(x)$ under the natural map $W \to Y$ consists of only finitely many points; let S denote the semilocal scheme of Y at these points. Since $F(S) = F_{Zar}(S)$ by 21.12, there is an open $V_x \subset Y$ such that $f_x = f|_{V_x} \in F_{Zar}(V_x)$ lies in the subgroup $F(V_x) \subseteq F_{Zar}(V_x)$. By 21.14, there is a neighborhood U_x of x such that $\mathcal{W}$ restricts to a finite correspondence $\mathcal{W}_x$ from U_x to V_x. Let $\mathcal{W}^*(f)_x$ denote the image of f_x under $\mathcal{W}_x^* : F(V_x) \to F(U_x) \subseteq F_{Zar}(U_x)$.

Uniqueness. Suppose that $F \to F_{Zar}$ is a morphism of presheaves with transfers. Given $\mathcal{W} \in Cor(X, Y)$ and $f \in F_{Zar}(Y)$, it suffices to show that $\mathcal{W}^*(f) \in F_{Zar}(X)$ is uniquely defined in some neighborhood of any point x. The construction above shows that the image of $\mathcal{W}^*(f)$ in $F_{Zar}(U_x)$ must equal $\mathcal{W}^*(f)_x$, which is defined using only the sheaf structure on F_{Zar} and the transfer structure on F.

Existence. Fix $\mathcal{W} \in Cor(X, Y)$ and $f \in F_{Zar}(Y)$. In the construction above, we produced a neighborhood U_x of every point $x \in X$, an open set V_x in Y so that $f_x = f|_{V_x}$ belongs to the subgroup $F(V_x)$ of $F_{Zar}(V_x)$, and considered the image $\mathcal{W}^*(f)_x = \mathcal{W}_x^*(f_x)$ of f_x in $F(U_x) \subseteq F_{Zar}(U_x)$.

We first claim that the $\mathcal{W}^*(f)_x$ agree on the intersections $U_{xx'} = U_x \cap U_{x'}$. The element $\mathcal{W}^*(f) \in F_{Zar}(X)$ will then be given by the sheaf axiom (see figure 21.1).

Pick two points $x, x' \in X$ and set $U_{xx'} = U_x \cap U_{x'}$, $V_{xx'} = V_x \cap V_{x'}$. Since $W \times_X U_x$ lies in $U_x \times V_x$ for all x (by 21.13), it follows that $W \times_X U_{xx'}$ lies in $U_{xx'} \times (V_x \cap V_{x'})$. Hence there is a finite correspondence $\mathcal{W}_{xx'}$ from $U_{xx'}$ lifting both $\mathcal{W}_x$ and $\mathcal{W}_{x'}$ in the sense of 21.14. That is, the middle square commutes in figure 21.1.

$$\begin{tikzcd}
F_{Zar}(Y) \arrow[d] & & & F_{Zar}(X) \arrow[d] \\
\prod F_{Zar}(V_x) \arrow[d] & \prod F(V_x) \arrow[l, hook'] \arrow[r, "\mathcal{W}^*_{U_x}"] \arrow[d] & \prod F(U_x) \arrow[r, hook] \arrow[d] & \prod F_{Zar}(U_x) \arrow[d] \\
\prod F_{Zar}(V_{xx'}) & \prod F(V_{xx'}) \arrow[l, "\text{into}"'] \arrow[r, "\mathcal{W}^*_{xx'}"] & \prod F(U_{xx'}) \arrow[r] & \prod F_{Zar}(U_{xx'})
\end{tikzcd}$$

Figure 21.1: The transfer map for F_{Zar}

Fix $x \in X$ and choose $V \subset Y$, $f_V \in F(V)$ and U_x as above. Because F is separated we have $F(V) \subset F_{Zar}(V)$, so the element $f_V \in F(V)$ is well defined. Given a dense $V_0 \subset V$, the map $F(V) \to F(V_0)$ sends f_V to f_{V_0}, because $F_{Zar}(V) \subset F_{Zar}(V_0)$ by 21.8. Given $U_0 \subset U_x$, the proof of 21.14 shows that the the canonical lift $\mathcal{W}_{U_0} \in Cor(U_0, V)$ is the composition of the inclusion $U_0 \subset U$ with the canonical lift $\mathcal{W}_U \in Cor(U, V)$. Hence $F_{Zar}(U_x) \to F_{Zar}(U_0)$ sends the element $\mathcal{W}^*(f)_x$ to the image of f_{V_0} under $F(V_0) \to F(U_0) \subset F_{Zar}(U_0)$.

It is now easy to check using 21.8 that the maps $\mathcal{W}^*$ are additive and give F_{Zar} the structure of a presheaf with transfers. $\square$

Proof of 21.2. We have to prove that $F_{Zar} = F_{Nis}$. Let F and F'' denote the kernel and cokernel presheaves of $F \to F_{Nis}$, respectively. By 13.1, they are presheaves with transfer whose associated Nisnevich sheaf is zero. Since sheafification is exact, it suffices to show that $F'_{Zar} = F''_{Zar} = 0$. That is, we may assume that $F_{Nis} = 0$.

By 21.1 and 21.15, F_{Zar} is also a homotopy invariant presheaf with transfers. Since $F_{Nis} = (F_{Zar})_{Nis}$, we may assume that $F = F_{Zar}$, i.e., that F is a Zariski sheaf. Therefore it suffices to show that $F(S) = 0$ for every local scheme S of a smooth variety X. Let S be the local scheme associated to a point x of X.

By 12.7, it suffices to show that, for any upper distinguished square

$$Q: \quad \begin{tikzcd}
B \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[d, "f"] \\
A \arrow[r, "i"] & X
\end{tikzcd}$$

as in 12.5, the square $F(Q \times_X S)$ is a pull-back.

Shrinking X around x, we may suppose by 11.17 that X is affine and fits into a standard triple $(\bar{X}, X_\infty, Z)$ with $A = X - Z$. Shrinking Y around the finite set $f^{-1}(x)$, we may also suppose by 11.17 that Y is affine, and fits into a standard triple so that Q comes from a finite morphism of standard triples in the sense of 20.1. Hence 20.11 implies that the map $F(MV(Q)) \to F(MV(Q \times_X S))$ is chain homotopic to zero.

$$\begin{tikzcd}
0 \arrow[r] & F(X) \arrow[r] \arrow[d] & F(A) \oplus F(Y) \arrow[r] \arrow[d] & F(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(S) \arrow[r] & F(A \cap S) \oplus F(Y \times_X S) \arrow[r] & F(B \times_X S) \arrow[r] & 0
\end{tikzcd}$$

Taking the limit over smaller and smaller neighborhoods X of x, we see that $F(MV(Q \times_X S))$ is exact. But then $F(Q \times_X S)$ is a pullback square, as claimed. □

Lecture 22

Contractions

Let F be a homotopy invariant presheaf. Define a new presheaf F_{-1} (known as the *contraction* of F in the literature) by the formula:

$$F_{-1}(X) = \mathrm{coker}\left(F(X \times \mathbb{A}^1) \to F(X \times (\mathbb{A}^1 - 0))\right).$$

For $r > 1$ we define F_{-r} to be $(F_{1-r})_{-1}$.

Since the inclusion $t = 1 : X \hookrightarrow X \times (\mathbb{A}^1 - 0) \subset X \times \mathbb{A}^1$ is split by the projection $X \times \mathbb{A}^1 \to X$, we have a canonical decomposition $F(X \times (\mathbb{A}^1 - 0)) \cong F(X) \oplus F(X)_{-1}$. Hence, F_{-1} is also homotopy invariant, and if F is a sheaf then so is F_{-1}. Here are some examples of this construction.

Example 22.1. If $F = \mathcal{O}_X^*$ then $F_{-1} = \mathbb{Z}$, because $\mathcal{O}^*(X \times (\mathbb{A}^1 - 0)) = \mathcal{O}^*(X) \times \{t^n\}$ for every integral X. By 4.1, there is a quasi-isomorphism $\mathbb{Z}(1)_{-1} \simeq \mathbb{Z}[-1]$.

More generally, the higher Chow groups $CH^i(-, n)$ are homotopy invariant (see p. 149) and their contractions are given by the formula:

$$CH^i(X, n)_{-1} \cong CH^{i-1}(X, n-1). \tag{22.1.1}$$

This follows from the the Localization Sequence (see p. 149):

$$CH^{i-1}(X, n) \xrightarrow{(t=0)_*} CH^i(X \times \mathbb{A}^1, n) \to CH^i(X \times (\mathbb{A}^1 - 0), n) \to CH^{i-1}(X, n-1),$$

which is split as above by the pullback along $t = 1$ (using 19.13).

Theorem 19.1 allows us to rewrite the formula in (22.1.1) as:

$$H^{m,i}(X, \mathbb{Z})_{-1} = \mathbb{H}^m_{Zar}(X, \mathbb{Z}(i))_{-1} \cong \mathbb{H}^{m-1}_{Zar}(X, \mathbb{Z}(i-1)) = H^{m-1,i-1}(X, \mathbb{Z}).$$

This yields the formula $\mathbb{Z}(i)_{-1} \simeq \mathbb{Z}(i-1)[-1]$ in the derived category, and in **DM**.

Example 22.2. We will see in the next lecture (in 23.1 and 23.8) if F is a homotopy invariant Zariski sheaf with transfers then $H^n(-,F)$ is homotopy invariant and $H^n_{Zar}(-,F)_{-1} \cong H^n_{Zar}(-,F_{-1})$.

Example 22.3. Suppose that $1/n \in k$, and let M be a locally constant n-torsion sheaf, such as μ_n. The argument of 22.1 applied to étale cohomology, shows that

$$H^m_{\acute{e}t}(X, M \otimes \mu_n)_{-1} \cong H^{m-1}_{\acute{e}t}(X, M).$$

Exercise 22.4. Let $\mathcal{U}$ be the standard covering of $X \times (\mathbb{A}^n - 0)$ by $U_1 = X \times (\mathbb{A}^1 - 0) \times \mathbb{A}^{n-1}$, ..., $U_n = X \times \mathbb{A}^{n-1} \times (\mathbb{A}^1 - 0)$. If F is homotopy invariant and $n \geq 2$, show that $\check{H}^0(\mathcal{U}, F) \cong F(X)$, $\check{H}^{n-1}(\mathcal{U}, F) \cong F_{-n}(X)$, and that $\check{H}^r(\mathcal{U}, F) = 0$ for all other r.

Now suppose that F is a Zariski sheaf, and that its cohomology groups are also homotopy invariant. Show that, for all m and $n > 0$, the cohomology with supports satisfies:

$$H^m_{X \times \{0\}}(X \times \mathbb{A}^n, F) \cong H^{m-n}(X, F)_{-n}.$$

Hint: Use the Čech spectral sequence $\check{H}^p(\mathcal{U}, H^q F) \Rightarrow H^{p+q}(X \times (\mathbb{A}^n - 0), F)$.

Proposition 22.5. *Let F be a homotopy invariant presheaf with transfers. Then $(F_{Nis})_{-1} \cong (F_{-1})_{Nis}$.*

Proof. By 13.1 and 21.3, F_{Nis} is a homotopy invariant sheaf with transfers. By inspection, the natural map $(F_{-1})_{Nis} \to (F_{Nis})_{-1}$ is a morphism of presheaves with transfers. By 11.2 (applied to the kernel and cokernel), it suffices to show that $F_{-1}(S) = (F_{Nis})_{-1}(S)$ when $S = \operatorname{Spec} E$ for a field E. The left side is $F(\mathbb{A}^1_E - 0)/F(\mathbb{A}^1_E)$ by definition, while the right side equals $F_{Nis}(\mathbb{A}^1_E - 0)/F_{Nis}(\mathbb{A}^1_E)$. These are equal by 21.4 and 21.2. □

In the rest of this lecture, we will compare F_{-1} to various sheaves $F_{(Y,Z)}$, which we now define.

Definition 22.6. Given a closed embedding $i : Z \hookrightarrow Y$, and a presheaf F, we define a Nisnevich sheaf $F_{(Y,Z)}$ on Z as follows. Let $K = K_{(Y,Z)}$ denote the presheaf cokernel of $F \to j_* j^* F$, where $j : V \hookrightarrow Y$ is the complement of Z. That is, $K(U)$ is the cokernel of $F(U) \to F(U \times_Y V)$ for all U. We set $F_{(Y,Z)} = (i^* K)_{Nis}$.

Since sheafification is exact, there is a canonical exact sequence of sheaves

$$F_{Nis} \to (j_* j^* F)_{Nis} \to i_* F_{(Y,Z)} \to 0. \tag{22.6.1}$$

Example 22.7. If $Z = \{z\}$ is a closed point on Y, then the value at Z of $F_{(Y,Z)}$ is the cohomology with supports, $H^1_Z(Y, F_{Nis})$. Indeed, if S is the Hensel local scheme of Y at Z then $F_{(Y,Z)}(Z)$ is the cokernel of $F_{Nis}(S) \to F_{Nis}(S - Z, F)$, i.e., $H^1_Z(S, F_{Nis})$. But this equals $H^1_Z(Y, F_{Nis})$ by excision [Har77, Ex.III.2.3]. Similarly, we have $H^n(-, F)_{(Y,Z)} \cong H^{n+1}_Z(Y, F)$ for $n > 0$. This follows from excision and the exact sequence

$$H^{n-1}(S, F) \to H^{n-1}(U, F) \to H^n_z(S, F) \to 0.$$

Example 22.8. Fix a Nisnevich sheaf F and consider the presheaf $H^n(-, F)$. We claim that if $n > 0$ then

$$H^n(-, F)_{(Y,Z)} = i^* R^n j_*(F).$$

Indeed, in 22.6.1 we have $H^n(-, F)_{Nis} = 0$, and $R^n j_*(F)$ is the sheaf on Y associated to the presheaf $j_* j^* H^n(-, F) = j_* H^n(-, F|_V)$. Hence

$$i_* H^n(-, F)_{(Y,Z)} \cong (j_* j^* H^n(-, F))_{Nis} = R^n j_*(F).$$

Now apply i^* and observe that $i^* i_*$ is the identity.

Example 22.9. Let $i : S \hookrightarrow S \times \mathbb{A}^1$ be the embedding $i(s) = (s, 0)$, with complement $S \times (\mathbb{A}^1 - 0)$. By definition, $F_{-1}(U) = K(U \times \mathbb{A}^1)$ where the cokernel presheaf K is defined in 22.6. The adjunction yields a natural map from $K(U \times \mathbb{A}^1)$ to $i_* i^* K(U \times \mathbb{A}^1) = i^* K(U)$. That is, we have a natural morphism of sheaves on S:

$$(F_{-1})_{Nis} \to F_{(S\times\mathbb{A}^1, S\times 0)}.$$

Proposition 22.10. *Let F be a homotopy invariant presheaf with transfers. Then $(F_{-1})_{Nis}|_S \cong F_{(S\times\mathbb{A}^1, S\times 0)}$ for all smooth S.*

Proof. We need to compare F_{-1} and $j_* j^* F/F$ at a sufficiently small neighborhood of any point s of any smooth affine S. We will use the standard triple $T = (\mathbb{P}^1_S \to S, S \times \{\infty\}, S \times \{0\})$, which is split over $S \times \mathbb{A}^1$ by 11.12. For each affine neighborhood U of $S \times 0$ in $S \times \mathbb{A}^1$, set $T_U = (\mathbb{P}^1_S, \mathbb{P}^1_S - U, S \times \{0\})$.

We claim that by shrinking S we can make T_U into a standard triple. At issue is whether or not $(\mathbb{P}^1_S - U) \cup (S \times \{0\})$ lies in an affine open subscheme of $\mathbb{P}^1_S$. Since the fiber U_s over s is open in $\mathbb{P}^1_s$, there is an affine open $V \subset \mathbb{P}^1_k$ so that $s \times_k V$ contains both $\{0\}$ and the finite set $\mathbb{P}^1_s - U_s$. Hence the complements of U and $S\times V$ in $\mathbb{P}^1_S$ intersect in a closed subset, disjoint from the fiber $\mathbb{P}^1_s$. Since $\mathbb{P}^1_S$ is proper over S, we may shrink S about s (keeping S affine) to assume that the complements are disjoint. Hence the affine $S\times V$ contains the complement $\mathbb{P}^1_S - U$ as well as $S\times\{0\}$, as claimed.

Now the identity on $\mathbb{P}^1_S$ is a finite morphism of standard triples $T_U\to T$ in the sense of 20.1 by 20.2. Setting $U_0=U-(S\times\{0\})$, the square Q coming from this is:

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & U \arrow[d] \\ S\times(\mathbb{A}^1-0) \arrow[r, "j"] & S\times\mathbb{A}^1 \end{tikzcd}$$

By the standard triples theorem 20.6 applied to $Q'=Q$, the complex $F(MV(Q))$ is split exact:

$$0\to F(S\times\mathbb{A}^1)\to F(S\times(\mathbb{A}^1-0))\oplus F(U)\to F(U_0)\to 0.$$

Since F is homotopy invariant, this implies that $F(U)\to F(U_0)$ is injective and that $F_{-1}(S)\cong F(U_0)/F(U)$. Since $j: S\times(\mathbb{A}^1-0)\hookrightarrow S\times\mathbb{A}^1$ has $j_*j^*F(U)=F(U_0)$, the right side is $j_*j^*F/F(U)$. Passing to the limit over U and S, we get the statement. □

Lemma 22.11. *Let $f: Y\to X$ be an étale morphism and Z a closed subscheme of X such that $f^{-1}(Z)\to Z$ is an isomorphism. Then for every presheaf F:*

$$F_{(X,Z)}\xrightarrow{\cong}F_{(Y,f^{-1}(Z))}.$$

Proof. Since this is to be an isomorphism of Nisnevich sheaves, we may assume that X is Hensel local, and that Z is not empty. Then Y is Hensel semilocal; the assumption that $f^{-1}(Z)\cong Z$ implies that Y is local and in fact $Y\cong X$. In this case the two sides are the same, namely $F(X-Z)/F(X)\cong F(Y-Z)/F(X)$. □

Lemma 22.11 uses the Nisnevich topology in a critical way. For the Zariski topology, the corresponding result requires F to be a homotopy invariant presheaf with transfers, and may be proven along the same lines as 22.10; see [CohTh, 4.13].

Theorem 22.12. *Let $i : Z \to X$ be a closed embedding of smooth schemes of codimension* 1*, and F a homotopy invariant presheaf with transfers. Then there exists a covering $X = \cup U_\alpha$ and isomorphisms on each $U_\alpha \cap Z$:*

$$F_{(U_\alpha, U_\alpha \cap Z)} \cong (F_{-1})_{Nis}.$$

That is, for each α there is an exact sequence of Nisnevich sheaves on U_α:

$$0 \to F_\alpha \to j_{\alpha *} j_\alpha^* F_\alpha \to i_*(F_{-1})_{Nis} \to 0.$$

Here $F_\alpha = (F|_{U_\alpha})_{Nis}$ and j_α denotes the inclusion $U_\alpha \cap (X - Z) \hookrightarrow U_\alpha$.

Moreover, for every smooth T we also have isomorphisms on $(U_\alpha \cap Z) \times T$:

$$F_{(U_\alpha \times T, (U_\alpha \cap Z) \times T)} \cong (F_{-1})_{Nis}.$$

Proof. We have to show that every smooth pair (X, Z) of codimension one is locally like $(S \times \mathbb{A}^1, S \times 0)$. If $\dim(Z) = d$ then, by shrinking X about any point (and writing X instead of U), we may find an étale map $f : X \to \mathbb{A}^{d+1}$ such that $Z \cong f^{-1}(\mathbb{A}^d)$.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ \mathbb{A}^d \arrow[r, hook] & \mathbb{A}^{d+1} \cong \mathbb{A}^d \times \mathbb{A}^1 \end{tikzcd}$$

By construction, $Z \times \mathbb{A}^1$ is étale over $\mathbb{A}^d \times \mathbb{A}^1$. Form the pullback $X' = X \times_{\mathbb{A}^{d+1}} Z \times \mathbb{A}^1$ and note that both $X' \to X$ and $X' \to Z \times \mathbb{A}^1$ are étale with $Z' = Z \times_{\mathbb{A}^d} Z$ lying above Z and $Z \times 0$, respectively. Since $Z' \to Z$ is étale and has a canonical section Δ, we can write $Z' = \Delta(Z) \coprod W$. Setting $X'' = X - W$, both $X'' \to X$ and $X'' \to Z \times \mathbb{A}^1$ are étale, with $\Delta(Z)$ the inverse image of Z and $Z \times 0$, respectively. Applying lemma 22.11 twice and then 22.10, we obtain the required isomorphisms of Nisnevich sheaves on Z:

$$F_{(X,Z)} \xleftarrow{\cong} F_{(X'', \Delta(Z))} \xrightarrow{\cong} F_{(Z \times \mathbb{A}^1, Z \times 0)} \cong (F_{-1})_{Nis}.$$

To see that the sequence of sheaves is exact, we only need to observe that F_α injects into $j_* j^* F_\alpha$ by lemma 21.8, since $F_\alpha = (F|_{U_\alpha})_{Zar}$ by 21.2.

In order to prove the final assertion, it suffices to replace Z, X and $\mathbb{A}^d$ with $Z \times T$, $X \times T$ and $\mathbb{A}^d \times T$ in the above argument. ∎

Porism 22.13. *The same proof shows that if $Z \to X$ is a closed embedding of smooth schemes of codimension r, then locally $F_{(X,Z)} \cong F_{(Z\times\mathbb{A}^r, Z\times 0)}$.*

Example 22.14. Let M be a locally constant n-torsion étale sheaf and consider $F(X) = H^1(X, M \otimes \mu_n)$. By 22.3, $(F_{-1})_{Nis} \cong M$. By [Mil80, p. 243], we also have $F_{(X,Z)} \cong M$. In this case, the isomorphisms $F_{(X,Z)} \cong (F_{-1})_{Nis}$ of 22.12 hold for any cover of X.

Lecture 23

Homotopy Invariance of Cohomology

We finally have all the tools to prove 13.7 which we restate here for the convenience of the reader.

Theorem 23.1. *Let k be a perfect field and F a homotopy invariant presheaf with transfers. Then $H^n_{Nis}(-, F_{Nis})$ is a homotopy invariant presheaf (with transfers) for every n.*

Proof. It suffices to prove that the $H^n_{Nis}(-, F_{Nis})$ are homotopy invariant, since we already know that they are presheaves with transfers from 13.4. We shall proceed by induction on n. The case $n = 0$ was completed in Theorem 21.3, so we know that F_{Nis} is homotopy invariant. Hence, we may assume that $F = F_{Nis}$.

Consider $X \times \mathbb{A}^1 \xrightarrow{\pi} X$. Since $\pi_* F(U) = F(U \times \mathbb{A}^1) \cong F(U)$, we have $\pi_* F = F$. By induction we know that $R^q \pi_* F = 0$ for $0 < q < n$. By theorem 23.2 below, $R^n \pi_* F = 0$ as well. Hence the Leray spectral sequence

$$H^p_{Nis}(X, R^q \pi_* F) \Rightarrow H^{p+q}_{Nis}(X \times \mathbb{A}^1, F)$$

collapses enough to yield $H^n_{Nis}(X, F) \cong H^n_{Nis}(X \times \mathbb{A}^1, F)$. That is, the presheaf $H^n_{Nis}(-, F)$ is homotopy invariant. $\square$

We have thus reduced the proof of 23.1 to the following theorem. Recall from [EGA4, 17.5] that the Hensel local scheme $\mathrm{Spec}(R)$ of a smooth variety at some point is *formally smooth*, i.e., geometrically regular.

Theorem 23.2. *Let k be a perfect field, and F a homotopy invariant Nisnevich sheaf with transfers such that $R^q\pi_*F = 0$ for $0 < q < n$. If S is a formally smooth Hensel local scheme over k, then $H^n_{Nis}(S \times \mathbb{A}^1, F) = 0$.*

The requirement that k be perfect comes from the following fact (see [EGA0, 19.6.4]): if k is perfect, every regular local k-algebra is formally smooth over k.

Proof. We will proceed by induction on $d = \dim(S)$. If $d = 0$ then $S = \operatorname{Spec}(K)$ for some field K; in this case, $H^n_{Nis}(S \times \mathbb{A}^1, F) = H^n_{Nis}(\mathbb{A}^1_K, F) = 0$ by 21.7. Here we have used exercise 2.9 to regard F as a homotopy invariant presheaf with transfers over K.

If $\dim(S) > 0$, and U is any proper open subscheme, then $R^q\pi_*F|_U = 0$ for $0 < q \leq n$, by induction on d. Thus the canonical map $\pi|_U^* : H^n_{Nis}(U, F) \to H^n_{Nis}(U \times \mathbb{A}^1, F)$ is an isomorphism, and its inverse is induced by the restriction $s|_U$ of the zero section $s : S \to S \times \mathbb{A}^1$ to U. From the diagram

$$\begin{tikzcd} H^n_{Nis}(S \times \mathbb{A}^1, F) \arrow[r, "j^*"] \arrow[d, "s^*"'] & H^n_{Nis}(U \times \mathbb{A}^1, F) \arrow[d, "s|_U^*"', "\cong"] \\ 0 = H^n_{Nis}(S, F) \arrow[r] & H^n_{Nis}(U, F) \end{tikzcd}$$

we see that the top map j^* is zero for all such U.

Now $S = \operatorname{Spec}(R)$ for a regular local ring $(R, \mathfrak{m})$; choose $r \in \mathfrak{m} - \mathfrak{m}^2$ and set $Z = \operatorname{Spec}(R/r)$, $U = S - Z$. Because Z is regular and k is perfect, Z is formally smooth over k. For this choice, the map j^* is an injection by proposition 23.3 below. Hence the source $H^n_{Nis}(S \times \mathbb{A}^1, F)$ of j^* must be zero. $\square$

Proposition 23.3. *Let k be a perfect field and S the Hensel local scheme of a smooth scheme X at some point. Let U be the complement of a smooth divisor Z on S. Under the inductive assumption that $R^q\pi_*F = 0$ for all $0 < q < n$, the following map is a monomorphism:*

$$H^n_{Nis}(S \times \mathbb{A}^1, F) \to H^n_{Nis}(U \times \mathbb{A}^1, F).$$

Proof. Let i and j denote the inclusions of $Z \times \mathbb{A}^1$ and $U \times \mathbb{A}^1$ into $S \times \mathbb{A}^1$ respectively. Regarding F as a sheaf on $S \times \mathbb{A}^1$, the map in question factors as:

$$H^n_{Nis}(S \times \mathbb{A}^1, F) \xrightarrow{\tau} H^n_{Nis}(S \times \mathbb{A}^1, j_*j^*F) \xrightarrow{\eta} H^n_{Nis}(U \times \mathbb{A}^1, j^*F).$$

We first show that the right-hand map η is injective. This will follow from 23.4 below, once we have shown that $R^q j_* F = 0$ for $0 < q < n$. The inductive assumption implies that $H^q(F)$ is a homotopy invariant presheaf with transfers. Since $q > 0$ we have $H^q(F)_{Nis} = 0$. Now see from 22.5 that $(H^q(F)_{-1})_{Nis} \cong (H^q(F)_{Nis})_{-1} = 0$. By 22.8 and 22.12 (with $T = \mathbb{A}^1$) we have

$$R^q j_* F \cong i_* H^q(F)_{(S\times\mathbb{A}^1, Z\times\mathbb{A}^1)} \cong i_*(H^q(F)_{-1})_{Nis} = 0.$$

We now prove that the left-hand map τ is injective as well. Since F is a homotopy invariant presheaf with transfers, F injects into j_*j^*F by lemma 21.8. By 22.6, there is a short exact sequence of Nisnevich sheaves on $S \times \mathbb{A}^1$:

$$0 \to F \to j_*j^*F \to i_*F_{(S\times\mathbb{A}^1, Z\times\mathbb{A}^1)} \to 0.$$

Since S is local, theorem 22.12 (with $T = \mathbb{A}^1$) implies that $F_{(S\times\mathbb{A}^1, Z\times\mathbb{A}^1)} \cong F_{-1}$ on $Z \times \mathbb{A}^1$. Consider the associated long exact sequence in cohomology.

$$\begin{aligned} H^{n-1}(S \times \mathbb{A}^1, j_*j^*F) \to H^{n-1}(Z \times \mathbb{A}^1, F_{-1}) \xrightarrow{\partial} & \\ H^n(S \times \mathbb{A}^1, F) \to H^n(S \times \mathbb{A}^1, j_*j^*F) &\to H^n(Z \times \mathbb{A}^1, F_{-1}) \end{aligned}$$

It suffices to show that the map $H^{n-1}(S\times\mathbb{A}^1, j_*j^*F) \to H^{n-1}(Z\times\mathbb{A}^1, F_{-1})$ is onto. If $n > 1$, this follows from the homotopy invariance of F_{-1} and the fact that Z is Hensel local:

$$H^{n-1}(Z \times \mathbb{A}^1, F_{-1}) \cong H^{n-1}(Z, F_{-1}) = 0.$$

If $n = 1$, we argue as follows. Since F and F_{-1} are homotopy invariant, the two left horizontal maps are isomorphisms in the commutative diagram:

$$\begin{tikzcd} F(U) \arrow[r, "\cong"] \arrow[d, "22.12"', "\text{onto}"] & F(U \times \mathbb{A}^1) \arrow[d] & H^0(S \times \mathbb{A}^1, j_*j^*F) \arrow[l, "="'] \arrow[d] \\ F_{-1}(Z) \arrow[r, "\cong"] & F_{-1}(Z \times \mathbb{A}^1) & H^0(Z \times \mathbb{A}^1, F_{-1}). \arrow[l, "="'] \end{tikzcd}$$

The left vertical map is onto by 22.12, because S is local. It follows that the right vertical map is onto, as desired. □

Lemma 23.4. *Let G be any sheaf on $U \times \mathbb{A}^1$ such that $R^q j_* G = 0$ for $0 < q < n$. Then the canonical map $H^n(X \times \mathbb{A}^1, j_*G) \to H^n(U \times \mathbb{A}^1, G)$ is an injection.*

Proof. Consider the Leray spectral sequence

$$H^p(X \times \mathbb{A}^1, R^q j_* G) \Longrightarrow H^{p+q}(U \times \mathbb{A}^1, G).$$

Using the assumption on the vanishing of the $R^q j_* G$, it is easy to see that there is a short exact sequence:

$$0 \to H^n(X \times \mathbb{A}^1, j_*G) \to H^n(U \times \mathbb{A}^1, G) \to H^0(X \times \mathbb{A}^1, R^n j_* G). \qquad \square$$

We have now completed the proof of homotopy invariance of the cohomology sheaves, which was promised in lecture 13 (as theorem 13.7).

For the rest of this lecture, we fix a homotopy invariant Zariski sheaf with transfers F over a perfect field k. Because we have proven theorem 13.7, we may use proposition 13.8, which says that $H^*_{Zar}(X, F) \cong H^*_{Nis}(X, F)$. We will sometimes supress the subscript and just write $H^*(X, F)$.

Corollary 23.5. *If S is a smooth semi-local scheme over k and F is a homotopy invariant sheaf with transfers, then for all $n > 0$:*

- *$H^n(S, F) = 0$;*
- *$H^n(S \times T, F) = 0$ for every open subset T of $\mathbb{A}^1_k$.*

Proof. (Cf. 13.8.) By 23.1, each $H^n(-, F)$ is a homotopy invariant presheaf with transfers. If E is the field of fractions of S, then $H^n(\operatorname{Spec} E, F) = 0$ for $n > 0$ because $\dim E = 0$. By 11.1, this implies that $H^n(S, F) = 0$.

Now $H^n(X) = H^n(X \times T, F)$ is also a homotopy invariant presheaf with transfers by 23.1, and $H^n(S)$ injects into $H^n(\operatorname{Spec} E) = H^n(\operatorname{Spec}(E) \times T, F)$ by 11.1. By 2.9 and 21.7, this group vanishes for $n > 0$. □

Example 23.6. Let $(R, \mathfrak{m})$ be a discrete valuation ring containing k, with field of fractions E and residue field $K = R/\mathfrak{m}$. Setting $S = \operatorname{Spec} R$ and $Z = \operatorname{Spec} K$, theorem 22.12 yields $F_{(S,Z)} \cong F_{-1}$ and an exact sequence of

Nisnevich sheaves on S, $0 \to F \to j_*F \to i_*F_{-1} \to 0$. Since $H^1_{Nis}(S,F) = 0$ by 23.5, the Nisnevich cohomology sequence yields the exact sequence:

$$0 \to F(\operatorname{Spec} R) \to F(\operatorname{Spec} E) \to F_{-1}(\operatorname{Spec} K) \to 0.$$

More generally, if R is a semilocal principal ideal domain with maximal ideals $\mathfrak{m}_i$, the same argument (using 23.5) yields an exact sequence:

$$0 \to F(\operatorname{Spec} R) \to F(\operatorname{Spec} E) \to \oplus_i F_{-1}(\operatorname{Spec} R/\mathfrak{m}_i) \to 0.$$

Exercise 23.7. If X is a smooth curve over k, show that $F_{-1}(x) \cong H^1_x(X,F)$ for every closed point $x \in X$. Conclude that there is an exact sequence

$$0 \to F(X) \to F(\operatorname{Spec} k(X)) \to \bigoplus_{x\in X} F_{-1}(x) \to H^1_{Zar}(X,F) \to 0.$$

Proposition 23.8. *Let k be a perfect field and F a homotopy invariant Zariski sheaf with transfers. Then $H^n(-,F)_{-1} \cong H^n(-,F_{-1})$ for all smooth X. That is, there is a natural isomorphism:*

$$H^n_{Zar}(X \times (\mathbb{A}^1 - \{0\}), F) \cong H^n_{Zar}(X,F) \oplus H^n_{Zar}(X,F_{-1}).$$

Proof. Write T for $\mathbb{A}^1 - \{0\}$ and consider the projection $\pi : X \times T \to X$. Let S be the local scheme at a point x of X. The stalk of $R^q\pi_*F$ at x is $H^q(S \times T, F)$, which vanishes for $q > 0$ by 23.5. Therefore the Leray spectral sequence degenerates to yield $H^n(X \times T, F) \cong H^n(X, \pi_*F)$. But $\pi_*F \cong F \oplus F_{-1}$ by the definition of F_{-1}. □

Example 23.9. Let F be a homotopy invariant Zariski sheaf with transfers. Combining proposition 23.8 with 23.1 and 22.4, we get the formula:

$$H^n_{Z\times\{0\}}(Z \times \mathbb{A}^r, F) \cong H^{n-r}(Z, F_{-r}).$$

If $Z = \operatorname{Spec}(K)$ for a field K, this shows that $H^n_{\{0\}}(\mathbb{A}^r_K, F)$ vanishes for $n \neq r$, while the value of $H^r_{\{0\}}(\mathbb{A}^r_K, F)$ at $\operatorname{Spec}(K)$ is $F_{-r}(\operatorname{Spec}(K))$.

Lemma 23.10. *Let S be a d-dimensional regular local scheme over a perfect field k. If F is a homotopy invariant sheaf with transfers and Z is the closed point of S, then $H^n_Z(S,F)$ vanishes for $n \neq d$, while $H^d_Z(S,F) \cong F_{-d}(Z)$.*

Proof. Since the case $d = 0$ is trivial, and $d = 1$ is given in example 23.6, we may assume that $d > 1$. Write U for $S - Z$. Since $F(S)$ injects into $F(U)$ by 11.1, $H^0_Z(S, F) = 0$. For $n > 0$, we may use $H^{n-1}(-, F)$, which is a homotopy invariant presheaf with transfers by 23.1. By 22.11 and two applications of 22.7, we have

$$H^n_Z(S, F) \cong H^{n-1}(-, F)_{(S,Z)} \cong H^{n-1}(-, F)_{(Z\times\mathbb{A}^d, Z\times 0)} \cong H^n_{Z\times 0}(Z \times \mathbb{A}^d, F).$$

By 23.9, this group vanishes for $n \neq d$, and equals $F_{-d}(Z)$ if $n = d$. □

If z is a point of X with closure Z, and A is an abelian group, let $(i_z)_*(A)$ denote the constant sheaf A on Z, extended to a sheaf on X.

Theorem 23.11. *Let X be smooth over k, and F a homotopy invariant Zariski sheaf with transfers. Then there is a canonical exact sequence of Zariski sheaves on X:*

$$0 \to F \to \coprod_{\operatorname{codim} z=0} (i_z)_*(F) \to \coprod_{\operatorname{codim} z=1} (i_z)_*(F_{-1}) \to \dots \to \coprod_{\operatorname{codim} z=r} (i_z)_*(F_{-r}) \to \dots$$

Proof. It suffices to assume that X is local with generic point x_0 and closed point x_d, and construct the exact sequence

$$0 \to F(S) \to F(x_0) \to \coprod_{\operatorname{codim} z=1} (F_{-1}(z)) \to \dots \to \coprod_{\operatorname{codim} z=r} (F_{-d}(z)) \to \dots \to F(x_d) \to 0.$$

When $\dim(X) = 1$ this is 23.6, so we may assume that $d = \dim(X) > 1$. For any $r \leq d$, let $H^n(X^r, F)$ denote the direct limit of the groups $H^n(X - T, F)$ with $\operatorname{codim}(T) > r$. For any Zariski sheaf F, and $r > 0$, the direct limit (over T and all Z of codimension r) of the long exact cohomology sequences $H^*_Z(X - T, F) \to H^*(X - T, F) \to H^*(X - Z - T, F)$ yields an exact sequence

$$0 \to \coprod_{\substack{\operatorname{codim} z\\ =r}} H^0_z(X_z, F) \to F(X^r) \to F(X^{r-1}) \to \coprod_{\substack{\operatorname{codim} z\\ =r}} H^1_z(X_z, F) \to H^1(X^r, F) \dots$$

Each X_z is an r-dimensional local scheme. Hence the groups $H^n_z(X_z, F)$ vanish except for $n = r$ by 23.10, and $H^r_z(X_z, F) \cong F_{-r}(z)$. For $r > 0$ this yields:

$$F(X) \cong F(X^{d-1}) \cong \cdots F(X^r) \cong \cdots \cong F(X^1);$$

$$0 = H^r(X, F) \cong H^r(X^{d-1}, F) \cong \cdots \cong H^r(X^{r+1}, F);$$

and (since X^0 is a point)

$$0 = H^r(X^0, F) \cong H^r(X^1, F) \cong \cdots \cong H^r(X^{r-1}, F).$$

Using these, we get exact sequences:

$$0 \to F(X) \to F(x_0) \to \coprod_{\operatorname{codim} z=1} H^1_z(X_z, F) \to H^1(X^1, F) \to 0;$$

and (for $0 < r \leq d$)

$$0 \to H^{r-1}(X^{r-1}, F) \to \coprod_{\operatorname{codim} z=r} H^r_z(X_z, F) \to H^r(X^r, F) \to 0.$$

Splicing these together (and using 23.10) yields the required exact sequence.

□

Remark 23.12. Since the sheaves $(i_z)_*(F_{-r})$ are flasque, theorem 23.11 gives a flasque resolution of the sheaf F. Taking global sections yields a chain complex which computes the cohomology groups $H^n(X, F)$. This shows that the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{\operatorname{codim} x=p} H_z^{p+q}(X, F) \Longrightarrow H^{p+q}(X, F)$$

degenerates, with $E_2^{p,0} = H^p(X, F)$ and $E_2^{p,q} = 0$ for $q \neq 0$.

Bibliography

[ABS87] R. MacPherson A. Beilinson and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J. **54** (1987), 679–710.

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics.

[BG73] K. S. Brown and S. M. Gersten, *Algebraic K-theory as generalized sheaf cohomology*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 266–292. Lecture Notes in Math., Vol. 341.

[BL94] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, Preprint, to appear in Inventiones Math., 1994.

[Blo86] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304.

[Blo94] ______, *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), no. 3, 537–568.

[BT73] H. Bass and J. Tate, *The Milnor ring of a global field*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972) (Berlin), Springer, 1973, pp. 349–446. Lecture Notes in Math., Vol. 342.

[FS00] E. M. Friedlander and A. Suslin, *The spectral sequence relating algebraic k-theory to motivic cohomology*, Preprint. http://www.math.uiuc.edu/K-theory/0432, 2000.

[Ful75] W. Fulton, *Rational equivalence on singular varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 147–167.

[Ful84] ______, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[GD71] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique 1*, Springer-Verlag, New York, 1971.

[GL00] T. Geisser and M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), no. 3, 459–493.

[GM88] S. Gel′fand and Yu. Manin, *Metody gomologicheskoi algebry. Tom 1*, "Nauka", Moscow, 1988, Vvedenie v teoriyu kogomologii i proizvodnye kategorii. [Introduction to the theory of cohomology, and derived categories], English translation published in 1996 by Springer-Verlag.

[Gro57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

[EGA0] ______, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas.*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 20, Reprinted in [GD71].

[EGA1] ______, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228, Reprinted in [GD71].

[EGA3] ______, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents.*, Inst. Hautes Études Sci. Publ. Math. (1961,1963), no. 11,17.

[EGA4] ______, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas.*, Inst. Hautes Études Sci. Publ. Math. (1964-1967), no. 20,24,28,32.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Jou73] J.-P. Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 293–316. Lecture Notes in Math., Vol. 341.

[KS86] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Amer. Math. Soc., Providence, RI, 1986, pp. 255–331.

[Lev94] M. Levine, *Bloch's higher chow groups revisited*, K-theory (Strasbourg, 1992), 1994, pp. 235–320.

[Lev98] ______, *Mixed motives*, American Mathematical Society, Providence, RI, 1998.

[Lic84] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Springer, Berlin, 1984, pp. 127–138.

[Lic94] ______, *Motivic complexes*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 303–313.

[Mac71] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971.

[Man68] Ju. Manin, *Correspondences, motifs and monoidal transformations*, Mat. Sb. (N.S.) **77 (119)** (1968), 475–507.

[May01] J. P. May, *The additivity of traces in triangulated categories*, Adv. Math. **163** (2001), no. 1, 34–73.

[Mil70] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344.

[Mil80] J. Milne, *Étale cohomology*, Princeton University Press, Princeton, N.J., 1980.

[Nis89] Ye. A. Nisnevich, *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory*, Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), Kluwer Acad. Publ., Dordrecht, 1989, pp. 241–342.

[NS89] Yu. Nesterenko and A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.

[Rec71] D. Rector, *K-theory of a space with coefficients in a (discrete) ring*, Bull. Amer. Math. Soc. **77** (1971), 571–575.

[Ric89] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[Ser65] J.-P. Serre, *Algèbre locale. Multiplicités*, Springer-Verlag, Berlin, 1965.

[SGA4] *Théorie des topos et cohomologie étale des schémas*, Springer-Verlag, Berlin, 1972-1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269, 270, 305.

[Sus82] A. Suslin, *Mennicke symbols and their applications in the K-theory of fields*, Proceedings of a Conference held at Oberwolfach, June 16–20, 1980 (R. K. Dennis, ed.), Springer-Verlag, Berlin, 1982, pp. 334–356.

[Sus83] ______, *On the K-theory of algebraically closed fields*, Invent. Math. **73** (1983), no. 2, 241–245.

[SV96] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties*, Invent. Math. **123** (1996), no. 1, 61–94.

[SV00] ______, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The Arithmetic and Geometry of Algebraic Cycles (B. Gordon, J. Lewis, S. Müller-Stach, S. Saito, N. Yui, ed.), Nato ASI series C, vol. 548, Kluwer, 2000, pp. 117–189.

[Swa62] R. G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277.

[Swa72] ______, *Some relations between higher K-functors*, J. Algebra **21** (1972), 113–136.

[VSF00] V. Voevodsky, A. Suslin and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000.

[Ver96] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[Voe02] V. Voevodsky, *Cancellation theorem*, Preprint. http://www.math.uiuc.edu/K-theory/541, 2002.

[Wei89] C. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Amer. Math. Soc., Providence, RI, 1989, pp. 461–488.

[Wei94] ———, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.

[Whi78] George W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

References to the articles in [VSF00].

[RelCy] A. Suslin and V. Voevodsky, *Relative Cycles and Chow Sheaves*, in [VSF00], pp. 10–86.

[CohTh] V. Voevodsky, *Cohomological Theory of Presheaves with Transfers*, in [VSF00], pp. 87–137.

[BivCy] E. M. Friedlander and V. Voevodsky, *Bivariant Cycle Cohomology*, in [VSF00], pp. 138–187.

[TriCa] V. Voevodsky, *Triangulated Categories of Motives Over a Field*, in [VSF00], pp. 188–254.

[HigCh] A Suslin, *Higher Chow Groups and Étale Cohomology*, in [VSF00], pp. 239–254.